A CLASSICAL LIMIT

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Sets of Kronecker products of finite dimensional irreducible representations of Lie algebras of types \( B_n, C_n \) and \( D_n \), possessing the following properties are pointed out: a) For a given set, the highest weight of one of the factors in the Kronecker product is a multiple \( m\Lambda \) of a given fundamental weight with \( m \) positive and integer; b) For a given set, the Clebsch-Gordan decomposition is the same for any \( m \) and depends only on \( m \). Only sets with Clebsch-Gordan series lengths three and four have been considered. For a given set of this type, the limit for \( m \to \infty \) of the minimal polynomial equations satisfied by the Hannabuss-operator associated with the Kronecker products lead to equations (of degree three and four) for equivariant operators \( K \), defined previously; these equations are equivalent to classical tensorial identities.

I. INTRODUCTION

It has been long been known that realizations of a semi-simple Lie algebra \( L \) satisfy specific polynomial identities, both for classical realizations (i.e. for Poisson bracket realizations) as well as for quantum realizations (i.e. for linear representations). In previous works (Refs 1-3) we pointed out that such identities results by equating to zero well defined tensors in the symmetric algebra \( S(L) \) of \( L \) and in the enveloping algebra \( U(L) \), respectively. We therefore used the term “tensorial identities”. These tensors are transforming under symmetric subrepresentations of Kronecker powers of the adjoint representation of \( L \). All second-degree symmetric tensors in \( S(L) \) and in \( U(L) \) have been determined (Refs. 2,3). As is well known [4] the last ones result from the first ones by symmetrization with respect to order in the products. For the quantum realizations of the semisimple Lie algebras of types \( A_n, B_n, C_n \) and \( D_n \), all finite-dimensional linear representations which satisfy the “tensorial identities” of degree two have been determined [3]. By using, in addition, a method due to Hannabuss [5], who relates the tensorial identity satisfied by a representation \( \rho_\Lambda \) of \( L \) with highest weight \( \Lambda \) to its Kronecker product with other representations of \( L \), it has been proved that second-degree identities for finite-dimensional representations of classical semi-simple Lie algebras are related to specific Kronecker products \( \rho_\Lambda \otimes \rho_\Omega \), where \( \rho_\Lambda \) is a minuscule representation [6] and \( \rho_\Omega \) (its “partner”) is a representation whose highest weight \( \Omega \) is of the form \( \Omega = m\Lambda_\gamma \), where \( \Lambda_\gamma \) is a well-defined fundamental weight and \( m \) is an arbitrary positive integer. (These results are summarized in Table I of Ref. 7.) The \( m \) dependence of the “partner weights” \( \Omega = m\Lambda_\gamma \), \( m = 1, 2, \ldots \) suggests a possibility to derive a classical limit for the identities associated with the product \( \rho_\Lambda \otimes \rho_\Omega \), by taking the limit \( m \to \infty \). In the limit \( m \to \infty \) the (second-degree) identities satisfied by the Hannabuss operators \( O_{\Lambda,\Omega} \) associated with the pairs of highest weights listed in Table I (of the Ref. 7) go into second-degree identities for the operators \( K \) (defined in Refs. 2, 8), which are associated with \( \rho_\Lambda \) and with a Poisson bracket realization. The existence of this classical limit is based on the possibility to associate Poisson bracket...
realizations of a Lie algebra $L$ to group orbits through the highest weights of finite dimensional representations of $L$. The aim of the present paper is to point out identities of degrees higher than two satisfied by linear representations. To do that we need to point out sets of Kronecker products possessing the following properties: a) All terms of a given set of Kronecker products have Clebsch-Gordan (CG) decompositions of equal length. b) For a given set, the CG decomposition is the same for any $m$ and depends only on $m$. These properties ensure that the minimal polynomials satisfied by the Hannabuss operators have the same expression and the same dependence on $m$. Sets of Kronecker products with these properties can be constructed by using an extension of a theorem due to Feingold [9]. (cf. also Ref. 10). We have identified in this way, for the classical Lie algebras of types $B_n, C_n$ and $D_n$, sets of Kronecker products whose CG series are of lengths three and four. By taking the limit, for $m \to \infty$, of the minimal polynomials associated with these Kronecker products we obtained identities satisfied by the corresponding $K$ – operators. The classical limits of the minimal polynomials associated with each set of products are summarized in the last section.

2. INVARIANT AND EQUIVARIANT OPERATORS

a) The Hannabus operator.

Let $\rho_{\Lambda}$ be a finite-dimensional representation with highest weight $\Lambda$ of the semisimple $n-$ dimensional Lie algebra $L$. We denote by $e_k$, $k = 1, \ldots, n$ a basis in $L$ and by $e^k$, $k = 1, \ldots, n$ a dual basis in $L$ with respect to the Cartan-Killing bilinear form: $(e_k, e^j) = \delta_{kj}$. Then the second-order Casimir operator of the representation $\rho_{\Lambda}$ is defined by:

$$C_2(\Lambda) = \sum_{k=1}^n \rho_{\Lambda}(e_k) \otimes \rho_{\Lambda}(e^k)$$

(2.1)

and the Hannabus operator of two representations $\rho_{\Lambda}$ and $\rho_{\Omega}$ is defined by [5] (cf. also Ref. 11):

$$O_{\Lambda,\Omega} = \sum_{k=1}^n \rho_{\Lambda}(e_k) \otimes \rho_{\Omega}(e^k)$$

(2.2)

The operator $O_{\Lambda,\Omega}$ has been defined by Hannabus [5]; cf. also Ref. [11].

Properties of $O_{\Lambda,\Omega}$:

1. The operator $O_{\Lambda,\Omega}$ is invariant under adjoint action.

2. $O_{\Lambda,\Omega}$ can be expressed in terms of Casimir operators of the representations $\rho_{\Lambda} \otimes \rho_{\Omega}$, $\rho_{\Lambda}$ and $\rho_{\Omega}$.

3. $O_{\Lambda,\Omega}$ commutes $\rho_{\Lambda} \otimes \rho_{\Omega}$.

4. Let :

$$\rho_{\Lambda} \otimes \rho_{\Omega} = \bigoplus_{\Xi \in CG(\Lambda,\Omega)} \rho_{\Xi}$$

(2.3)

be the CG decomposition of product $\rho_{\Lambda} \otimes \rho_{\Omega}$ (whose set of weights is denoted by $CG(\Lambda,\Omega)$). The minimal polynomial satisfied by the operator $O_{\Lambda,\Omega}$ has the expression:
where $2\delta$ is the sum of the positive roots of the Lie algebra $L$. The degree of Eq. (2.4) is clearly equal with the length of the $CG$ series of $\rho_\Lambda \otimes \rho_\Omega$. The matrix elements of the characteristic equation (obtained by equating to zero the minimal polynomial) taken between states belonging to one of the representations ($\rho_\Lambda$, say) provide identities satisfied by the representation $\rho_\Omega$. The invariance property of the operator $O_{\Lambda,\Omega}$ implies that the matrix elements of the minimal polynomial are tensors.

### a) The moment-like mapping.

Let us replace in Eq. (2.2) one of the quantum realization by a classical one, e.g. write instead of the generator $\rho_\Omega(e_j)$ of the linear representation $\rho_\Omega$ the generator $f_{e_j}(m)$ of the Poisson bracket realization on a symplectic $G$-manifold $M$, ($e_j \in L, m \in M$). We obtain, in this way, a mapping:

$$K : M \rightarrow End V_\Lambda$$

defined by:

$$K(m) = \sum_{j=1}^n f_{e_j}(m) \rho_\Lambda (e^j), \ (m \in M)$$

Assuming that $S_\Lambda(g)$ is the representation of $G$ ($L = Lie(G)$) acting in $V_\Lambda$, the following property of the operator $K$ can be proved:

$$K(g.m) = S_\Lambda(g^{-1})K(m)S_\Lambda(g) \quad (g \in G)$$

i.e. the mapping $K$ is equivariant. An immediate consequence is that any polynomial $P(K)$ is equivariant. Polynomial relations satisfied by the operator $K$ are equivalent to sets of polynomial relations satisfied by the generators of the Poisson bracket realization.

### 3. THE CLASSICAL LIMIT

To transform Eq. (2.4) satisfied by $O_{\Lambda,\Omega}$ into an equation for an operator $K$ of type (2.6) we remind that the Poisson brackets realizations on the co-adjoint orbit through a highest weight $\Lambda$ is generated by the covariant symbols, i.e.:

$$f_s(\xi) = f_s(Ad(g)^*\Lambda) = f_{Ad_\Lambda (\xi)}(\Lambda) = \langle v_\Lambda | \rho_\Lambda (Ad(g))v_\Lambda \rangle$$

where $|v_\Lambda>$ denotes the highest weight vector of the representation $\rho_\Lambda$. Reminding that:

$$\rho_\Lambda (Ad^* (g) x) = S^{-1}_\Lambda (g) \rho_\Lambda (x) S_\Lambda (g)$$

and that:

$$S_\Lambda (g) \otimes S_\Omega (g) O_{\Lambda,\Omega} S_\Lambda (g^{-1}) \otimes S_\Omega (g^{-1}) = O_{\Lambda,\Omega}$$

we define the operator $K_{\Lambda,\Omega}$ associated with the Hannabuss operator $O_{\Lambda,\Omega}$ by:
whence, with:
\[ \xi = \text{Ad}^\ast (g) \Lambda \]

\[ K_{\Lambda,\Omega}(\xi) = \sum_{i=1}^{n} f_{e_i}(\text{Ad}^\ast (g) \Lambda) \rho_{\Omega}(e^i) = < v_{\Lambda} \left| (S_{\Lambda}(g) \otimes I)O_{\Lambda,\Omega}(S_{\Lambda}(g^{-1}) \otimes I) \right| v_{\Lambda} > = \]

\[ < v_{\Lambda} \left| (I \otimes S_{\Omega}(g^{-1}))O_{\Lambda,\Omega}(I \otimes S_{\Omega}(g)) \right| v_{\Lambda} > = S_{\Omega}(g^{-1})K_{\Lambda,\Omega}(\Lambda)S_{\Omega}(g) \]

and the equivariance property of the operator \( K \) is proved. As:
\[ < v_{m\Lambda} \left| \rho_{m\Lambda}(x) \right| v_{m\Lambda} > = m\Lambda(x) \]

we have:
\[ K_{m\Lambda,\Omega}(\xi) = mK_{\Lambda,\Omega}(\xi) \]

As proved in Ref. 12, the principal part of:
\[ < v_{m\Lambda} \left| O_{m\Lambda,\Omega} \right| v_{m\Lambda} > \]

behaves like \( m^q K_{\Lambda,\Omega} \). In particular, for \( q = 2 \), second-degree equations for \( O_{m\Lambda,\Omega} \) become second-degree equations for operator denoted \( K_{\Omega}(\xi) = K_{\Lambda,\Omega}(\xi) \).

For the Hannabuss operators \( O_{m\Lambda,\Omega} \) associated with the products \( \rho_{m\Lambda} \otimes \rho_{\Omega} \) (with the \( CG \) series of length 2) the coefficients of the minimal polynomials:
\[ (O_{m\Lambda,\Omega} - k_1 I)(O_{m\Lambda,\Omega} - k_2 I) = 0 \]

where given in Ref. [8]. Hence the coefficients \( b = \lim_{m \to \infty} \frac{-1}{m}(k_1 + k_2) \) and \( c = \lim_{m \to \infty} \frac{1}{m^2}k_1k_2 \) of the second-degree equation satisfied by \( K_{\Omega} \) can be obtained directly [8].

## 4. FEINGOLD’S THEOREM

For the sake of transparency, starting with the present section we shall use for a representation of highest weight \( \Lambda \) the notation \((\Lambda)\) instead of \( \rho_{\Lambda} \). To obtain the classical limits for relations of degree three and four satisfied by the Hannabuss operator we proceeded in the following way: We found out the sets of Kronecker products of type \( (\Lambda) \otimes (m\Lambda) \) \((m = 1, 2, \ldots)\) with the following properties:

a) The products \( (\Lambda) \otimes (m\Lambda) \) \((m = 1, 2, \ldots)\) admits \( CG \) decompositions of equal lengths;

b) The terms of these decompositions have the same expression (in particular, the same \( m \)-dependence).

To construct sets of Kronecker products with these properties a generalization of a theorem due to Feingold [9] is of great help. To state it we introduce the following notations: \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are the simple roots of the complex semi-simple Lie algebra \( L \) of rank \( n \), \( \alpha = \frac{2\alpha^\vee}{(\alpha,\alpha)} \) is the coroot of the root \( \alpha \), \( \Lambda^\vee \) alis the set of
dominant weights corresponding to $\alpha_1, \alpha_2, \ldots, \alpha_n$, $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ are the fundamental weights in $\Lambda^+$, defined by:

$$<\Lambda, \alpha_j^\vee> = \delta_{ij}$$  \hspace{1cm} (4.1)

Simple roots and fundamental weights are related by:

$$\alpha_j = \sum_{j=1}^n <\alpha_j, \alpha_j^\vee> \Lambda_j$$  \hspace{1cm} (4.2)

where $<\alpha_i, \alpha_j^\vee>$ are elements of the Cartan matrix of $L$. For the Lie algebras $A_n, D_n, E_6, E_7, E_8$ the lengths of all roots are equal. For the Lie algebras $C_n, B_n, F_4, G_2$ the systems of roots divides in two subsystems: long roots and short roots. For each of these last Lie algebras, two roots are dominant weights: the highest long root $\alpha_{hl}$ and the highest short root $\alpha_{hs}$. Their expressions and as well as those of the corresponding coroots are displayed in Ref. [6]. With these prerequisites we shall now formulate Feingold’s theorem in the form stated in Ref. [10].

**THEOREM.** Let $L$ be a semi-simple Lie algebra of rank $n$. Let $\Lambda, \Omega$ be dominant weights of $L, (\Lambda, \Omega \in \Lambda^+)$, and let $(\Lambda), (\Omega)$ be the corresponding finite-dimensional irreducible representations. Assume that for the long (short) simple roots $\alpha_i$ of $L$ we have:

$$<\Omega, \alpha_i^\vee> \geq <\Lambda, \alpha_{hl(hs)}^\vee>$$  \hspace{1cm} (4.3)

Then, if:

$$(\Lambda) \otimes (\Omega) = \bigoplus_{\Gamma \in \Lambda'} m_\Gamma(\Gamma)$$  \hspace{1cm} (4.4)

($m_\Gamma$ is the multiplicity of the representation $(\Gamma)$) we have also:

$$(\Lambda) \otimes (\Omega + \Lambda_i) = \bigoplus_{\Gamma \in \Lambda'} m_\Gamma(\Gamma + \Lambda_i)$$  \hspace{1cm} (4.5)

( In Eq. (4.3) $\alpha_{hl(hs)}^\vee$ has the following meaning $\alpha_{hl(hs)}^\vee = \alpha_{hl}^\vee (\alpha_{hs}^\vee)$ if $\alpha_i$ is long (short). Using the same notation as in the theorem, the following corollary is immediate:

**COROLARY.** If for the long (short) simple roots $\alpha_i$ we have:

$$<\Omega, \alpha_i^\vee> \geq <\Lambda, \alpha_{hl(hs)}^\vee>$$  \hspace{1cm} (4.6)

and if:

$$(\Lambda) \otimes (\Omega) = \bigoplus_{\Gamma \in \Lambda'} m_\Gamma(\Gamma)$$  \hspace{1cm} (4.7)

then:

$$(\Lambda) \otimes (\Omega + \Lambda_i) = \bigoplus_{\Gamma \in \Lambda'} m_{\Gamma + \Lambda_i}(\Gamma + \Lambda_i)$$  \hspace{1cm} (4.8)

In particular, if:

$$(\Lambda) \otimes (\Lambda_i) = \bigoplus_{\Gamma \in \Lambda'} m_\Gamma(\Gamma)$$  \hspace{1cm} (4.9)

then:
5. RESULTS

To obtain classical limits for relations of degrees three and four satisfied by the Hannabuss operator we proceed in the following way: We find out pairs \((i, j)\) such that the Kronecker products of type \((\Lambda_i) \otimes (m\Lambda_j)\) \((m = 1, 2, \ldots)\) possess the following properties: a) For all values of \(m\) the products associated with a pair \((i, j)\) admit \(CG\) decompositions of equal lengths (three or four, respectively); b) Their \(CG\) decompositions are the same (modulo \(m\)) for any rank. Sets of Kronecker products with these properties are obtained using the above given generalization of a theorem of Feingold. Calculations have been done for algebras \(B_n, C_n\) and \(D_n\) for which we give below the Kronecker products and the classical limits in matrix form.

Lie algebras \(B_n\)

\[(\Lambda_i) \otimes (m\Lambda_j) = (\Lambda_i + m\Lambda_j) \oplus ((m-1)\Lambda_i + \Lambda_{i+1}) \oplus (\Lambda_{i-1} + (m-1)\Lambda_i); \quad (k \leq n-2)\]

Classical limit: \(K_{\Lambda_i}^3 - K_{\Lambda_i} = 0\).

\[(\Lambda_i) \otimes (m\Lambda_{n-1}) = (\Lambda_i + m\Lambda_{n-1}) \oplus ((m-1)\Lambda_{n-1} + 2\Lambda_n) \oplus (\Lambda_{n-2} + (m-1)\Lambda_{n-1})\]

Classical limit: \(K_{\Lambda_i}^3 - K_{\Lambda_i} = 0\).

\[(\Lambda_i) \otimes (m\Lambda_n) = (\Lambda_i + m\Lambda_n) \oplus (m\Lambda_n) \oplus (\Lambda_{n-1} + (m-2)\Lambda_n); \quad m \geq 2\]

Classical limit: \(K_{\Lambda_i}^3 - \frac{1}{4} K_{\Lambda_i} = 0\).

\[(m\Lambda_2) \otimes (\Lambda_n) = (m\Lambda_n + m\Lambda_2) \oplus (\Lambda_i + (m-1)\Lambda_2 + \Lambda_n) \oplus (\Lambda_n + (m-1)\Lambda_2)\]

Classical limit: \(K_{\Lambda_i}^3 - K_{\Lambda_i} = 0\).

\[(m\Lambda_k) \otimes (\Lambda_{k+1}) = (m\Lambda_k + \Lambda_{k+1}) \oplus ((m-1)\Lambda_k + \Lambda_{k+1}) \oplus ((m-1)\Lambda_1 + \Lambda_{k+1}) \oplus ((m-2)\Lambda_k + \Lambda_2); \quad (1 < k \leq n-2; \quad m > 2)\]

Classical limit: \(K_{\Lambda_i}^4 - K_{\Lambda_i}^2 = 0\).

\[(m\Lambda_{n-1}) \otimes (\Lambda_{n-1}) = (m\Lambda_{n-1} + \Lambda_{n-1}) \oplus ((m-1)\Lambda_1 + 2\Lambda_n) \oplus (\Lambda_{n-2} + (m-1)\Lambda_1) \oplus ((m-2)\Lambda_1 + \Lambda_{n-1}); \quad (m \geq 2)\]

Classical limit: \(K_{\Lambda_{n-1}}^4 - K_{\Lambda_{n-1}}^2 = 0\).

\[(m\Lambda_1) \otimes (2\Lambda_n) = (m\Lambda_1 + 2\Lambda_n) \oplus ((m-1)\Lambda_1 + 2\Lambda_n) \oplus (\Lambda_{n-1} + (m-1)\Lambda_1) \oplus ((m-2)\Lambda_1 + 2\Lambda_n); \quad (m \geq 2)\]

Classical limit: \(K_{\Lambda_1}^4 - K_{\Lambda_1}^2 = 0\).
A classical limit

Lie algebras $C_n$

$$(\Lambda_i) \otimes (m\Lambda_k) = (\Lambda_i + m\Lambda_k) \oplus ((m-1)\Lambda_k + \Lambda_{k+1}) \oplus (\Lambda_{k-1} + (m-1)\Lambda_k); \; (1 \leq k \leq n)$$

Classical limit: $K_{\Lambda_i}^3 - K_{\Lambda_i} = 0$.

$$(m\Lambda_i) \otimes (m\Lambda_n) = (m\Lambda_i + 2\Lambda_n) \oplus ((m-1)\Lambda_i + \Lambda_n) \oplus (\Lambda_{n-1} + (m-1)\Lambda_i) \oplus ((m-2)\Lambda_i + \Lambda_n); \; (m \geq 2)$$

Classical limit: $K_{\Lambda_i}^3 - K_{\Lambda_n} = 0$.

$$((\Lambda_2) \otimes (m\Lambda_n) = (m\Lambda_n + \Lambda_2) \oplus (\Lambda_1 + (m-1)\Lambda_n + \Lambda_{n-1}) \oplus (\Lambda_{n-2} + (m-1)\Lambda_n)$$

Classical limit: $K_{\Lambda_1}^3 - 4K_{\Lambda_i} = 0$.

$$(m\Lambda_i) \otimes (m\Lambda_k) = (m\Lambda_i + \Lambda_k) \oplus ((m-1)\Lambda_i + \Lambda_{k+1}) \oplus (\Lambda_{k-1} + (m-1)\Lambda_i) \oplus ((m-2)\Lambda_i + \Lambda_k); \; (1 < k \leq n-1; m \geq 2)$$

Classical limit: $K_{\Lambda_i}^3 - 2K_{\Lambda_i} = 0$.

$$((2\Lambda_1) \otimes (m\Lambda_n) = (2\Lambda_1 + m\Lambda_n) \oplus (\Lambda_1 + \Lambda_{n-1} + (m-1)\Lambda_n) \oplus (m\Lambda_n) \oplus (2\Lambda_1 + (m-2)\Lambda_n); \; (m \geq 2)$$

Classical limit: $K_{2\Lambda_i}^4 - 4K_{\Lambda_i}^2 = 0$.

$$(\Lambda_3) \otimes (m\Lambda_n) = (\Lambda_3 + m\Lambda_n) \oplus (\Lambda_2 + \Lambda_{n-1} + (m-1)\Lambda_n) \oplus (\Lambda_1 + \Lambda_{n-1} + (m-1)\Lambda_n) \oplus (\Lambda_{n-3} + (m-1)\Lambda_n)$$

Classical limit: $K_{\Lambda_i}^4 - 10K_{\Lambda_i}^2 + 9I = 0$.

$$(\Lambda_i) \otimes (m\Lambda_k + \Lambda_n) = (\Lambda_i + m\Lambda_k + \Lambda_n) \oplus (\Lambda_{k+1} + \Lambda_n + (m-1)\Lambda_k) \oplus (m\Lambda_k + \Lambda_{n-1}) \oplus (\Lambda_{k-1} + (m-1)\Lambda_k + \Lambda_n)$$

Classical limit: $K_{\Lambda_i}^4 - K_{\Lambda_i} = 0$.

Lie algebras $D_n$

$$(\Lambda_i) \otimes (m\Lambda_k) = (\Lambda_i + m\Lambda_k) \oplus ((m-1)\Lambda_k + \Lambda_{k+1}) \oplus (\Lambda_{k-1} + (m-1)\Lambda_k); \; (1 \leq k \leq n-2)$$

Classical limit: $K_{\Lambda_i}^3 - K_{\Lambda_i} = 0$.

$$(\Lambda_i) \otimes (m\Lambda_{n-2}) = (\Lambda_i + m\Lambda_{n-2}) \oplus ((m-1)\Lambda_{n-2} + \Lambda_{n-1} + \Lambda_n) \oplus (\Lambda_{n-3} + (m-1)\Lambda_{n-2})$$

Classical limit: $K_{\Lambda_i}^3 - K_{\Lambda_i} = 0$.

$$(m\Lambda_2) \otimes (\Lambda_n) = (\Lambda_n + m\Lambda_2) \oplus (\Lambda_1 + (m-1)\Lambda_2 + \Lambda_{n-1}) \oplus (\Lambda_n + (m-1)\Lambda_2)$$

Classical limit: $K_{\Lambda_i}^3 - K_{\Lambda_i} = 0$. 
Classical limit: $K^{4}_{\Lambda} - K^{2}_{\Lambda} = 0$.

The matrix identities $P(K^{4}_{\Lambda}) = 0$ obtained from the minimal polynomials satisfied by $O_{\Lambda_i, m\Lambda_j}$ are equations of co-adjoint orbits through the highest weight $\Lambda_j$ associated with the representation $\Lambda_i$. For the case $i = 1$ a simple explanation can be given for the form of these equations. Let us consider indeed the matrices $K_{\Lambda_1}$, which for the algebras $C_n$ and $D_n$ are of the form:

$$K_{\Lambda_1} = \begin{pmatrix} A & B \\ -C & -A' \end{pmatrix}$$

where $A$, $B$ and $C$ are $n \times n$ matrices and $B = B'$, $C = C'$ for $C_n$ algebras and $B = -B'$, $C = -C'$ for $D_n$ algebras. Thus, for the algebras $C_n$, the matrix $K_{\Lambda_1}$ has $n(2n+1)$ distinct matrix elements and for the algebras $D_n$, the matrix $K_{\Lambda_1}$ has $n(2n-1)$ distinct matrix elements. These are precisely the dimensions of the corresponding adjoint representations of these algebras. Indeed, for $C_n$ algebras we have $\dim(2\Lambda_1) = n(2n+1)$ and, for $D_n$ ($n \geq 4$), $\dim(\Lambda_2) = n(2n-1)$.

Let us now consider the matrix:

$$K^{2}_{\Lambda_1} = \begin{pmatrix} A^2 - BC & AB - BA' \\ -CA + A'C & (A^2 - BC)' \end{pmatrix}$$

The submatrices $AB = -B A'$ and $-CA + A'C$ are antisymmetric for the algebras $C_n$ and symmetric for the algebras $D_n$ i.e. they have $n(2n-1)$ and $n(2n+1)$ distinct matrix elements for $C_n$ and $D_n$ algebras, respectively. These are precisely the dimensions of representations $(0) \otimes (\Lambda_2) = ((\Lambda_1) \otimes (\Lambda_1))_{\text{antisym}}$ for $C_n$ and $(0) \otimes (2\Lambda_1) = ((\Lambda_1) \otimes (\Lambda_1))_{\text{sym}}$ for $D_n$. This phenomenon is general: odd powers of $K_{\Lambda_1}$ behaves like $K_{\Lambda_1}$, i.e. tensors of type $(2\Lambda_1)$ for $C_n$ and of type $(\Lambda_2)$ for $D_n$; even powers behaves like $K^{2}_{\Lambda_1}$, i.e. like tensors of type $(0) \otimes (\Lambda_1)$ for $C_n$ and of type $(2\Lambda_1)$ for $D_n$.

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