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POLYANALYTICITY AND POLYHARMONICITY

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The paper is dedicated to the centennial of two master precursors of the Romanian mathematics Nicolae Ciorãnescu and Miron Nicolescu. We survey the progress of some topics originated in their research.

From my high school years, I have been fascinated by the clarity of texts written by Prof. N.Ciorãnescu. Later, in the mid of 50’s, I was privileged to study Mathematical Analysis and Real Function Theory and after to carry out my Ph.D. thesis in 1964 with Prof. M. Nicolescu. Moreover, their contributions regarding the polyharmonicity formed a foundation of my research for more than one decade. Based on our achieve-ments, gathered in [27],we describe first the polyanalitycity in the case of functions of one complex variable, linked by the areolar derivative,and pass then to the polyharmonicity for functions in $\mathbb{R}^m$, $m \geq 3$, using matrix analysis or hypercomplex variables. We will emphasize the international follow up of some successful trends of the Romanian mathematical school.

The study of complex functions beyond those holomorphic has been an old concern of mathematicians. Also, the model equations of two-dimensional continuum mechanics required taking into account functions $f(z) = u(x,y) + \mathrm{i}v(x,y)$ of a complex variable $z = x + \mathrm{i}y$, which are not solutions of the Cauchy-Riemann system. Since its inception by D.Pompeiu in 1912, the *areolar derivative*

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + \mathrm{i}\,\frac{\partial f}{\partial y}\right] = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \mathrm{i}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

came into sight as a functional measuring the deviation from holomorphy of such a function, the holomorphic functions being the null-space of the areolar derivative. For the functions with continuous areolar derivative, he introduced in [31] the integral representation

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + T\left(\frac{\partial f}{\partial \bar{z}}\right)(z) = \begin{cases} f(z) & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{C}\setminus\Omega, \end{cases} \tag{1}$$

called the Cauchy-Pompeiu formula, where

$$Tg(z) = -\frac{1}{\pi}\iint_{\Omega}\frac{g(\zeta)}{\zeta - z}\,\mathrm{d}\xi\,\mathrm{d}\eta\,,\ \zeta = \xi + i\eta\,,$$

is the *Pompeiu operator*, Ω the domain interior to a rectifiable curve Γ in the complex plane $\mathbb{C}$.

The theory of areolar derivative became a traditional subject in the Romanian mathematics in the middle of the 20th century. Thus, it is resumed in the doctoral theses at Paris of G.Cãlugareanu and M.Nicolescu in 1928 and later in some studies by N.Ciorãnescu and M.Ghermãnescu. It is worth mentioning, that in his thesis at Paris in 1931, N.Teodorescu pointed out that the areolar derivative definition is independent of the existence of the partial derivatives of $u(x,y)$ and $v(x,y)$, anticipating in a way the concept of generalized derivative for integrable functions. We emphasize also that basic results of the areolar derivative theory were extended to the three-dimensional space by Gr.C. Moisil and N. Teodorescu in 1931.

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After World War II, among Romanian contributions to the areolar derivative and polyharmonique functions are those due to M. Roºculeþ, M.Nedelcu-Coroi, I.Elianu, S.Teleman, P.Caraman, C.Reicher-Haimovici, I.Toma and V.Iftimie.

It is worth noticing that, for a bounded measurable function g in Ω, N.Teodorescu [33] proved that *Tg(z)* is continuous in the whole plane, holomorphic outside $\overline{\Omega}$, vanishes at infinity, and

$$\frac{\partial\ Tg(z)}{\partial\ \overline{z}} = g(z).$$

Thus, we can regard T as an inverse operator of the areolar derivative. Furthermore, I.N. Vekua [34, pp.38] showed that T is a completely continuous linear operator from $L_p(\Omega)$, $p > 2$, into $C_a(\overline{\Omega})$, the Hölder function space with $a = \frac{p-2}{p}$.

Similarly, one can define the *mean derivative* by $\frac{\partial f}{\partial z} = \overline{\frac{\partial \overline{f}}{\partial \overline{z}}}$ and note that $\Delta = 2^2\frac{\partial^2 f}{\partial z \partial \overline{z}}$.

One of the natural generalization of analytic functions are the solutions of the iterative equation

$$\frac{\partial^{n+1}\Phi(z)}{\partial \overline{z}^{n+1}} = 0, \quad \Phi(z) = P(x,y) + \mathrm{i}Q(x,y),$$

called *polyanalytic functions* or *areolar polynomials of the n-th degree*. It is readily seen that they have the following representation

$$\Phi(z) = j_0(z) + \overline{z}j_1(z) + \ldots + \overline{z}^n j_n(z), \tag{2}$$

where $j_i(z)$, $i = 0,1,\ldots,n$, are holomorphic functions in a domain Ω, containing the origin. In 1959, we have shown that, under natural uniqueness conditions,

$$\begin{cases} \operatorname{Im}\left[j_0(0)\right] = 0, \\ \quad j_1(0) = 0, \quad \operatorname{Im}\left[j_1^{/}(0)\right] = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \quad j_n(0) = 0, \quad j_n^{/}(0) = 0, \ldots, \ j_n^{(n-1)}(0) = 0, \quad \operatorname{Im}\left[j_n^{(n)}(0)\right] = 0, \end{cases} \tag{3}$$

where Im [ ] and Re [ ] denote the imaginary and the real part of [ ], respectively, the above representation of Φ coincides with the *Almansi development* of the $(n+1)$ - th degree polyharmonic real functions $P(x,y) = \operatorname{Re}[\Phi(z)]$, i.e.,

$$P(x,y) = p_0(x,y) + r^2 p_1(x,y) + \ldots + r^{2n} p_n(x,y), \tag{4}$$

where $p_i(x,y)$, $i = 0,1,\ldots,n$, are harmonic functions in Ω and $r = |z|$. Under similar conditions (3), replacing Im by Re, we obtain the same development for $Q(x,y)$. Moreover, when conditions (3) hold, the holomorphic components of (2) have the form $j_k(z) = z^k\left[u_k + \mathrm{i}\ v_k\right]$, where u_k and v_k are harmonic functions and we can conversely pass from the development (4) to the representation (2). Thus, we have established.

THEOREM 1 ([23]). *In the complex plane let* Ω *be a star-like domain, with center at the origin. Under uniqueness conditions of type* (3)*, the representation* (2) *of a polyanalytic function of* n-th *degree is equivalent with the Almansi development* (4) *for polyharmonic functions of* (n+1)-th *degree of its real or imaginary part.*

Naturally, the polyanalytic functions inherit properties of holomorphic functions. The differences, similarities and generalizations are worked out in the monograph [2].

In a similar way to (2) one defines the areolar series

$$f(z) = \sum_{n=0}^{\infty} \overline{z}^n j_n(z), \tag{5}$$

where $j_k(z)$, $k = 0,1,2,...$, are holomorphic functions in Ω. It easy to check that this series is absolutely and uniformly convergent when f and its areolar derivatives of all orders are bounded in the domain Ω.

On the other side, N.Ciorãnescu [5] proved that any real analytic function $U(x,y)$ of two variables in a neighborhood Ω' of the origin admits an expansion in an uniformly convergent series of the form

$$U(x,y) = \sum_{k=0}^{\infty} r^{2k} u_k(x,y), \tag{6}$$

where $u_k(x,y)$, $k = 0,1,2,...$ are harmonic functions in Ω' and $r^2 = x^2 + y^2$. We have shown the following local correspondence:

THEOREM 2 [26]. *A necessary and sufficient condition for a real function to be analytic in a two-dimensional domain Ω is that it be either the real or imaginary part of an areolar series.*

Moreover, N.Ciorãnescu ([6],[7, pp.575-604]) extended the representation (6) for real analytic functions of several variables. In other words, the expansion (6) means a polyharmonic function of infinite degree. The exploration and illustration of the properties of polyharmonique functions of infinite degree and the interplay between the real and complex aspects were been started also in the 30's by N.Aronszajn and are reviewed in the book [1], with many reference to Romanian contributions.

Later in 1962, we proposed a new expansion of polyanalytic functions, by use of the iterate operator of Pompeiu, which is similar to an expansion of polyharmonique functions aplying the iterated Green's functions due to M.Nicolescu ([18], [20, p.326-335]). Indeed, denoting

$$K(t - z) = -\frac{1}{p(t - z)} \qquad \text{and} \qquad K^j(t - z) = \iint_\Omega K^{j-1}(t - z)K(z - z)\,\mathrm{d}x\,\mathrm{d}h, \quad j > 1,$$

we consider the Pompeiu iterated operators

$$T^j g(z) = \iint_\Omega K^j(z - z)g(z)\,\mathrm{d}x\,\mathrm{d}h \quad j = 1,2, ... ,n, \tag{7}$$

for a suitable function $g(z)$ so that the integral makes sense. Since

$$T^j g(z) = \iint_\Omega K(z - z)\mathrm{d}x\,\mathrm{d}h \iint_\Omega K^{j-1}(s - z)g(s)\,\mathrm{d}s\mathrm{d}t = T(T^{j-1}g)(z), \quad s = s + \mathrm{i}t,$$

we derive

$$\frac{\partial T^n g(z)}{\partial \overline{z}} = T^{n-1} g(z), \; . \; . \; . \; , \; \frac{\partial^n T g(z)}{\partial \overline{z}^n} = g(z).$$

Moreover, if the function *g(z)* is holomorphic in Ω, then

$$\frac{\partial^n T^n g(z)}{\partial \overline{z}^n} = 0.$$

Let us consider the formal polynomial

$$\Phi(z) = j_0(z) + Tj_1(z) + \ldots + T^n j_n(z), \tag{8}$$

where the functions j_i are holomorphic in Ω. Taking successively areolar derivatives, we get

$$\frac{\partial\, \Phi(z)}{\partial \overline{z}} = j_1(z) + \ldots + T^{n-1} j_n(z),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial^n \Phi(z)}{\partial \overline{z}^n} = j_n(z), \quad \text{and} \quad \frac{\partial^{n+1} \Phi(z)}{\partial \overline{z}^n} = 0,$$

we infer that $\Phi(z)$ is a polyanalytic function of degree n in Ω.

Likewise, since

$$\frac{\partial^k \Phi(z)}{\partial \overline{z}^k} = j_k(z) + T(j_{k+1}(z) + \ldots + T^{n-k-1} j_n(z)), \tag{9}$$

the functions j_k are determined by the value of $\dfrac{\partial^k \Phi}{\partial \overline{z}^k}$ on Γ by

$$j_k(z) = \frac{1}{2p\ i} \int_\Gamma \frac{\partial^k \Phi(z)}{\partial \overline{z}^k} \frac{\mathrm{d}z}{z - z}, \qquad k = 0,1,\ldots,n\,. \tag{10}$$

In this way, we have obtained the following structure result:

THEOREM 3 [25]. *Any polyanalytic function* $\Phi(z)$ *of degree* n in Ω *has the representation* (8), *where the holomorphic components are given by* (10).

By using Morera's theorem and equation (9), we derive

THEOREM 4. *Necessary and sufficient conditions that* $n+1$ *continuous functions* $y_k(z)$ *in* Ω *should represent an* n-th *degree polyanalytic function* $y_0(z) = \Phi(z)$ *and its areolar derivatives* $\dfrac{\partial^k \Phi(z)}{\partial \overline{z}^k} = y_k(z)$ *are that*

$$\int_g (y_{k-1}(z) - \mathrm{T} y_k(z))\, \frac{\mathrm{d}z}{z - z} = 0, \quad k = 1,2,\ldots,n,$$

for any closed simple rectifiable curve g *in* Ω, *enclosing a domain* w, *and any point* z *outside* $\overline{w}$.

We remark that the iterated operator (7) can be written in the concise form [3]:

$$T^j g(z) = \iint_\Omega K_j(z - z)\, g(z)\, \mathrm{d}x\, \mathrm{d}h \quad j = 1,2,\ldots,n\,,$$

where

$$K_j(z) = \frac{1}{(j-1)!\, p} \frac{\overline{z}^{n-1}}{z}\,.$$

For higher iteration operators a similar formula is given in [4], where a Pompeiu-like operator corresponding to the mean derivative has been also written and a representation (8) for the equations involving mixed derive-atives with respect to $\overline{z}$ and z has been established.

A version with expansion (8) for n - th order generalized analytic functions, when the derivative $\frac{\partial f}{\partial \bar{z}}$ was replaced by the operator $Lf = \frac{\partial f}{\partial \bar{z}} + \mathrm{A}f + \mathrm{B}\bar{f}$, (Vekua's equation, [34]), performed in [28].

Later on, analogous representations in the Euclidean spaces $\mathbb{R}^m$, $m \geq 3$, have been pointed out. First, we present briefly elements of matrix analysis, describing an extension of the monogenic conditions or Cauchy -Riemann equations.

Let S be a closed hypersurface with the interior normal $\vec{n}_M$ at each point M. Denote by V^+ and V^- the interior and exterior domains separated by S, respectively. Let $g = (g_1,...,g_m)$ be an m-tuple of constant square matrices of order $2s$ and let $\bar{g} = (\bar{g}_1,...,\bar{g}_m)$ be the vector of transposed matrices. We confine ourselves to matrix fields $\Psi = \{\Psi_{jk}\} \subset C^1(V)$ and we introduce the linear operators

$$\mathrm{D}\,\Psi = \sum_{\mathrm{j}=1}^{m} g_{\mathrm{j}} \frac{\partial \Psi}{\partial x_{\mathrm{j}}} \qquad \text{and} \qquad \bar{\mathrm{D}}\,\Psi = \sum_{\mathrm{j}=1}^{m} \bar{g}_{\mathrm{j}} \frac{\partial \Psi}{\partial x_{\mathrm{j}}}$$

called the spatial derivative and spatial co-derivative, respectively. To obtain $\bar{\mathrm{D}}\mathrm{D} = \mathrm{D}\bar{\mathrm{D}} = \mathrm{I}\Delta$, we impose the conditions

$$g_j\,\bar{g}_k + \bar{g}_j\,g_k = 2\,d_{jk}I, \qquad j,k = 1,2,...,m.$$

A simple example of such a set of fourth order matrices in $\mathbb{R}^3$ is [16] :

$$g_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Regarding the operator D, we have the Teodorescu-Moisil representation

$$\frac{1}{s_m}\int_S \frac{(\overrightarrow{MP}\cdot\bar{g})}{\overline{MP}^m}(n_M \cdot g)\Psi(M)\,\mathrm{d}s_M + \frac{1}{s_m}\int_{V^+} \frac{(\overrightarrow{QP}\cdot\bar{g})}{\overline{QP}^m} D\Psi(Q)\,\mathrm{d}w_Q = \begin{cases} \Psi(P) & if \quad P \in V^+ \\ 0 & if \quad P \in V^- \end{cases}, \tag{11}$$

where s_m is the area of the unit hypersphere and $(\mathrm{u}\cdot\mathrm{v})$ denotes the standard inner product in $\mathbb{R}^m$.

Solutions of the equations $\mathrm{D}\Psi = 0$ are called $(g)-$*monogenic*. In this framework, the integral representation (11) is actually an extension of the Cauchy-Pompeiu formula. A generalization of the Morera theorem for $(g)-$monogenic functions follows easily from the Gauss – Ostrogradskii formula.

On the other hand, the *Teodorescu-Moisil* integral operator

$$\Pi\Psi(P) = \frac{1}{s_m}\int_{V^+} \frac{(\overrightarrow{QP}\cdot\bar{g})}{\overline{QP}^m} D\Psi(Q)\,\mathrm{d}w_Q \tag{12}$$

extends to $\mathbb{R}^m$ all the properties of the Pompeiu operator. Moreover, we have defined in [27] the iteratives of the operator Π and proven that $(g)-$*polymonogenic* functions, i.e., solutions of the equation

$$D^{n+1}\Phi(P) = 0$$

admit a representation of the form

$$\Phi(P) = \Psi_0(P) + \Pi\Psi_1(P) + \Pi^2\Psi_2(P) + \ldots + \Pi^n\Psi_n(P),$$

where its $(g)-$monogenic components are determined by

$$\Psi_k(P) = \frac{1}{s_m}\int_S \frac{(\overrightarrow{MP}\cdot\overline{g})}{\overline{MP}^m}(n_M \cdot g)D^k\Phi(M)\,\mathrm{d}\,s_M\,, \quad k = 0,1,2,...,n.$$

In a similar way, we can prove a Morera theorem for $(g)-$polymonogenic functions.

The matrix equation $\mathrm{D}\Psi = 0$ includes Dirac's systems as well as other known systems. Using an elaborate technique, M.Nedelcu-Coroi [17] constructed some operators equivalent to $\frac{\overline{z}^n}{n!}$ to obtain an expansion like (2) for $(g)-$polymonogenic functions.

Functions of a hypercomplex variable provide an alternative basis for extending the underlying concepts of one complex variable functions to higher dimensional spaces. This approach was initiated by Gr.C.Moisil [14]. By replacing the algebra of square matrices with the quarternion field, we gain a weakening of the restrictions for the inversion of a singular integral equation system [24], generalizing a result, remarkable in 50's, due to A.V.Bicadze A splitting of the wave operator as product of two first-order complex quaternion operators in four-dimensional spaces has been also obtained. A theory of analytic functions for certain class of m-dimensional algebras over real field, such as Clifford algebras, Poincaré-Steklov problem, the Vilat equation as well as a complete description of the properties of integral operators of Pompeiu-Moisil-Teodorescu type have been investigated by V.Iftimie [10],[11]. These Romanian research, quoted by V.S.Fedorov, V.A. Gusev, N.T. Stelmasuk, K.D.Zatulovskaja in the 70's, some published in Romanian reviews, form intermediate steps between the original hypercomplex models in the theory of elasticity and hydrodynamics, initiated by Gr.C. Moisil in the 30's and 40's and the theory of generalized hypercomplex functions of R.P. Gilbert and his collaborators in the 70's and 80's and summarized in the monograph [9], containing many references to the Romanian researcher. We emphasize the recent research due to M. Roºculeþ, a disciple of Prof. N. Ciorãnescu, regarding areolar partial and volumetric derivatives on noncommutative algebras as well as the work of H.R.Malonek and G.Ren [12] on the extension of our results [23], [27] to Almansi-type expansions in Clifford algebras.

M.Nicolescu's investigations [21] concerning the iterative elliptic, parabolic and hyperbolic equations or, unified in an axiomatization of the generalized analyticity with respect to a linear operator over a normed algebras have been developed by many researchers, in particular, we note again the research of M. Roºculeþ (e.g. [32]).

Last but not the least, I would like to mention the contribution of Lilly-Jeanne Nicolescu [22] and Ioana Ciorãnescu [8] to the theory of areolar derivative.

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