AN EXISTENCE RESULT OF WEAK SOLUTIONS TO DIRICHLET PROBLEM FOR NONLINEAR SECOND ORDER SYSTEMS OF DIVERGENCE TYPE

Gheorghe.Gr. CIOBANU*, Gabriela SĂNDULESCU**

* Seminarul Matematic "Al. Myller", Universitatea “Al.I. Cuza” Iași
** Liceul “M. Eminescu” Iași

We obtain an existence result for the weak solutions in the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$, $p > 2$, to the Dirichlet problem for a nonlinear second order system of divergence type. In fact, it is proved that, in certain hypotheses, the operator naturally associated to the Dirichlet problem is a bounded and coercive Gårding operator [10]. We get a generalization of the results obtained in [4] for the Dirichlet problem of nonlinear elastostatics.

Key words: Sobolev spaces; weak solutions; Gårding operator.

4. SOME FEW PRELIMINARIES

A. The summation over repeated subscripts is understood and the notation $i = p, q$, where $p \leq q$ are integers, means that the index $i$ takes the values $p, p+1, \ldots, q$.

If $a, b \in \mathbb{R}^k$ then $a \cdot b$ is the standard scalar product on $\mathbb{R}^k$ and $|a|$ is the corresponding Euclidean norm of $a$. $\mathbb{M}^{n \times n}_{m \times m}$ denotes the linear space of matrices $A=(a_{ij})$ of elements $a_{ij} \in \mathbb{R}$, $i=1,m$, $j=1,n$. The application $(A,B) \mapsto \text{tr}(AB^T)$, $A,B \in \mathbb{M}^{n \times n}_{m \times m}$, is the standard inner product on $\mathbb{M}^{n \times n}_{m \times m}$ and $|A|$ is the corresponding norm of $A$.

B. Throughout this paper we suppose $\Omega \subset \mathbb{R}^m$ is a bounded Lipschitz domain ([1], [3], [5], [9]) with boundary $\partial \Omega$, and $\text{d}x$ denotes the Lebesgue measure on $\Omega$.

We use the notation [6] $L^p(\Omega, \mathbb{R}^m)$, $W^{1,p}(\Omega, \mathbb{R}^m)$, and $W_0^{1,p}(\Omega, \mathbb{R}^m)$, $p \in [1,\infty)$, for the Banach spaces of $\mathbb{R}^m$-valued functions $u=(u_1,\ldots,u_m):\Omega \to \mathbb{R}^m$, with components $u_k: \Omega \to \mathbb{R}$, $k=1,m$, belonging to Banach spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$, and $W_0^{1,p}(\Omega)$ respectively. $L^p(\Omega, \mathbb{R}^m)$ is a Banach space, separable for $p \in [1,\infty)$ and reflexive for $p \in (1,\infty)$, with respect to the norm

$$
\|u\|_{W_0^{1,p}(\Omega, \mathbb{R}^m)}:=\left(\int_\Omega |u|^p \text{d}x\right)^{1/p} \in [0,\infty), \quad u \in L^p(\Omega, \mathbb{R}^m).
$$

If $(u,v) \in L^p(\Omega, \mathbb{R}^m) \times L^{1/p'}(\Omega, \mathbb{R}^m)$, $p \in (1,\infty)$, $1/p + 1/p' = 1$, then the function $(u \cdot v)(x):=u(x) \cdot v(x)$, $x \in \Omega$, belongs to $L^{1}(\Omega)$ [7] and it holds the Hölder inequality

$$
\int_\Omega u \cdot v \text{d}x \leq \int_\Omega |u|^{1/p} |v|^{1/p'} \text{d}x \leq\|u\|_{L^p(\Omega, \mathbb{R}^m)}\|v\|_{L^{1/p'}(\Omega, \mathbb{R}^m)}.
$$

The dual of $L^p(\Omega, \mathbb{R}^m)$ is $L^{p'}(\Omega, \mathbb{R}^m)$, i.e. $(L^p(\Omega, \mathbb{R}^m))' = L^{p'}(\Omega, \mathbb{R}^m)$, and duality pairing on $L^p(\Omega, \mathbb{R}^m) \times L^{p'}(\Omega, \mathbb{R}^m)$ is defined by
The Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$ ([1]–[3], [5], [6], [9]) is separable for $p \in [1, \infty)$ and reflexive for $p \in (1, \infty)$, with respect to the norm
\[
\| u \|_{1,p} := \left( \int_{\Omega} |u|^p + |\nabla u|^p \, dx \right)^{1/p} = (\| u \|^p_{L^p} + \| \nabla u \|^p_{L^p})^{1/p} \in [0, \infty).
\]

Here $\nabla u$ is the distributional gradient of $u$, i.e.
\[
D_j u^i := \frac{\partial u^i}{\partial x^j},
\]
and $D_{ij} u$ is the $j$-th partial generalized derivative of $u$. $W^{1,p}(\Omega, \mathbb{R}^m)$ is a closed subspace of $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^m)$ and, in view of Poincare’s inequality ([2], [3], [6]), $W^{1,p}(\Omega, \mathbb{R}^m)$ is isomorphic to $\mathcal{W}^{1,p}(\Omega, \mathbb{R}^m)$ in the sense of the trace operator ([2], [3], [9]).

In our hypothesis on $\Omega$ we have the completely continuous imbedding ([2], [9])
\[
W^{1,p}(\Omega, \mathbb{R}^m) \subset L^p(\Omega, \mathbb{R}^m), \quad p \in (1, \infty),
\]
and for $p > 2$ the following continuous and dense imbeddings
\[
W^{1,p}(\Omega, \mathbb{R}^m) \subset L^p(\Omega, \mathbb{R}^m) \subset L^2(\Omega, \mathbb{R}^m) \subset W^{-1,p}(\Omega, \mathbb{R}^m),
\]
where $W^{-1,p}(\Omega, \mathbb{R}^m) := (W^{1,p}(\Omega, \mathbb{R}^m))^\prime$. If $p > 2$ then $p' \in (1, 2)$ and therefore
\[
L^p(\Omega, \mathbb{R}^m) \subset X(\Omega, \mathbb{R}^m) := L^p(\Omega, \mathbb{R}^m) \cap W^{-1,p}(\Omega, \mathbb{R}^m) \subset W^{-1,p}(\Omega, \mathbb{R}^m).
\]

The weak convergence in $L^p(\Omega, \mathbb{R}^m)$, denoted by $u_n \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^m)$, is defined by $\int_{\Omega} u_n \cdot v \, dx \to \int_{\Omega} u \cdot v \, dx$, $\forall v \in L^p(\Omega, \mathbb{R}^m)$, while the weak convergence in $W^{1,p}(\Omega, \mathbb{R}^m)$, denoted by $u_n \rightharpoonup u$ in $W^{1,p}(\Omega, \mathbb{R}^m)$, is equivalent with ([5], [6])
\[
u_n \to u \text{ and } D_j u_n \to D_j u, \quad i = 1, m, \text{ in } L^p(\Omega, \mathbb{R}^m),
\]
and implies the strong convergence $u_n \to u$ in $L^p(\Omega, \mathbb{R}^m)$ (Rellich Theorem [5]).

The quotient space $W^{1,p}(\Omega, \mathbb{R}^m)/W^{1,p}_0(\Omega, \mathbb{R}^m)$ is isomorphic to $W^{1,p'}(\partial \Omega, \mathbb{R}^m)$ in the sense of the trace operator ([2], [3], [9]).

C. The divergence operator on the set of mappings $S = (S_g): \Omega \to \mathbb{M}_{max}$, with $S_g \in W^{1,p}(\Omega)$ is defined by
\[
S \mapsto \text{div } S: \Omega \to \mathbb{R}^m, \quad (\text{div } S)_i := D_i S_g \in L^p(\Omega).
\]

D. Definition 1.1 Let $V = (V, \| \cdot \|)$ and $U = (U, \| \cdot \|_U)$, $V \subset U$, be two separable and reflexive Banach spaces. Suppose that $V$ is dense in $U$ and that the imbeding $V \subset U$ is completely continuous [1]. The operator $\Lambda: V \to V'$, where $V'$ is the topological dual of $V$, is said to be a Gårding operator [10] if $\Lambda(v) = F(v, v) \in V$ for $v \in V$, where the operator $F(\cdot, \cdot): V \times V \to V'$ satisfies the conditions:
An existence result of weak solutions to Dirichlet problem

For every \( w \in V \), \( F(\cdot, w) : V \to V' \) is hemicontinuous [8], i.e. the real function \( t \mapsto \langle v, F(u + tv, w) \rangle \in \mathbb{R} \), \( t \in \mathbb{R} \), is continuous for every \( u, v, w \in V \), \( \langle \cdot, \cdot \rangle \) being the pairing duality on \( V \times V' \).

There exists a continuous function \( \gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \), satisfying the condition
\[
\lim_{\theta \to 0} \theta \gamma(x, \theta y) = 0, \quad \forall x, y \in \mathbb{R}^+,
\]
such that \( \langle u - v, \Lambda(u) - F(v, u) \rangle \geq -\gamma(r, \|u - v\|_r) \), for every \( u, v \in B^r_f(0) = \{w \in V : \|w\| < r\} \).

If \( u_n \to u \) in \( V \), the conditions
\[
\liminf_{n \to \infty} \langle u_n - u, F(v, u_n) - F(v, u) \rangle \geq 0,
\]
\[
\liminf_{n \to \infty} \langle w, F(v, u_n) - F(v, u) \rangle \geq 0, \quad \forall u, v, w \in V.
\]
hold simultaneously.

One shows [10] that a bounded Gårding operator is a pseudomonotone operator [8].

**Definition 1.2** An operator \( \Lambda : V \to V' \) is said to be coercive [8] if
\[
\lim_{\|v\| \to \infty} \langle v, \Lambda(v) \rangle > \infty \quad \text{as} \quad \|v\| \to \infty. \tag{1.4}
\]

**Theorem 1.1** ([10]) If \( V \) is a reflexive and separable Banach space and \( \Lambda : V \to V' \) is a bounded and coercive Gårding operator then \( \Lambda \) is surjective, i.e. for every \( f \in V' \) the operator equation \( \Lambda(u) = f \) has at least a solution \( u \in V \).

## 2. SECOND ORDER SYSTEMS OF DIVERGENCE TYPE

We consider the following second order system of divergence type [9]
\[
-\text{div} S(u, \nabla u) - b(u, \nabla u) = f \tag{2.1}
\]
in the unknown function \( u = (u_1, \ldots, u_m) : \Omega \to \mathbb{R}^m \) from \( W^{1,p} (\Omega, \mathbb{R}^m) \), \( p > 2 \), where \( \Omega \subset \mathbb{R}^n \) is a Lipschitz domain and \( f = (f_1, \ldots, f_m) : \Omega \to \mathbb{R}^m \),
\[
x = (x_1, \ldots, x_n) \mapsto S(u, \nabla u) (x) := S(x, u(x), \nabla u(x)) \in M_{\text{mono}}, \quad x \in \Omega \subset \mathbb{R}^n, \tag{2.2}
\]
\[
x = (x_1, \ldots, x_n) \mapsto b(u, \nabla u(x)) := b(x, u(x), \nabla u(x)) \in \mathbb{R}^m, \quad x \in \Omega \subset \mathbb{R}^n, \tag{2.3}
\]
are given functions.

Now we present the restrictions imposed to mappings (2.2) and (2.3) for the solvability of the system (2.1) in \( W^{1,p} (\Omega, \mathbb{R}^m) \), \( p > 2 \).

(I) **Restrictions on \( S(\cdot, \cdot) \).**

a) For every \( (p, P) \in \mathbb{R}^m \times M_{\text{mono}} \), the mapping \( S(\cdot, p, P) : \Omega \to M_{\text{mono}} \) is (Lebesgue) measurable, i.e. its real components \( S_i(\cdot, p, P) : \Omega \to \mathbb{R} \), \( i = 1, m \), \( j = 1, n \) are measurable.

b) For almost every (a.e.) \( x \in \Omega \) the mapping \( S(x, \cdot, \cdot) : \mathbb{R}^n \times M_{\text{mono}} \to M_{\text{mono}} \) is Fréchet continuously differentiable. This implies that for a.e. \( x \in \Omega \) there exist the “partial derivatives” of \( S \) with respect to \( p \in \mathbb{R}^m \) and \( P \in M_{\text{mono}} \), i.e. the linear operator
\[
\begin{cases}
(p, P) \mapsto \frac{\partial S}{\partial p}(x, p, P) \in L(\mathbb{R}^m, \mathbb{M}_{\text{weak}}), \\
(p, P) \mapsto \frac{\partial S}{\partial P}(x, p, P) \in L(\mathbb{R}^m, \mathbb{M}_{\text{weak}})
\end{cases}
\]

which are continuous on \( \mathbb{R}^m \times \mathbb{M}_{\text{weak}} \) and are defined by

\[
q = (q_i) \mapsto \left( \frac{\partial S}{\partial p}(x, p, P) q_i \right) := \frac{\partial S}{\partial p_i}(x, p, P) q_i \in \mathbb{M}_{\text{weak}}, \quad q \in \mathbb{R}^m,
\]

\[
Q = (Q_{ij}) \mapsto \left( \frac{\partial S}{\partial P}(x, p, P) Q_{ij} \right) := \frac{\partial S}{\partial P_{ij}}(x, p, P) Q_{ij} \in \mathbb{M}_{\text{weak}}, \quad Q \in \mathbb{M}_{\text{weak}}.
\]

In (2.4) \( L(U, V) \) denotes the space of linear operators from the linear space \( U \) to the linear space \( V \).

c) For every \( (p, P) \in \mathbb{R}^m \times \mathbb{M}_{\text{weak}} \) the mappings

\[
\frac{\partial S}{\partial p_i}(\cdot, p, P), \frac{\partial S}{\partial P_{ij}}(\cdot, p, P): \Omega \to \mathbb{M}_{\text{weak}}, \quad i = \overline{1,m}, j = \overline{1,n},
\]

are measurable.  

d) Suppose that for every \( (x, p, P) \in \Omega \times \mathbb{R}^m \times \mathbb{M}_{\text{weak}} \) and \( i = \overline{1,m}, \ j = \overline{1,n} \), the following growth conditions hold:

\[
\begin{cases}
|S(x, p, P)| \leq \phi(x) + a^1 |p| + a^2 |P|, \\
|\frac{\partial S}{\partial p_i}(x, p, P)| \leq \phi_i(x) + a_i^1 |p| + a_i^2 |P|, \\
|\frac{\partial S}{\partial P_{ij}}(x, p, P)| \leq \phi_{ij}(x) + a_{ij}^1 |p| + a_{ij}^2 |P|,
\end{cases}
\]

(2.6)

where the real functions \( \phi, \phi_i, \phi_{ij} \) are from \( L^p(\Omega) \) and \( a^1, a^2; a_i^1, a_i^2; a_{ij}^1, a_{ij}^2 \) are positive constants independent of \( (x, p, P) \).

**Remark 2.1** We notice that the conditions \((I)_a \) and \((I)_b \) (it is required only the continuity of \( S(x, \cdot, \cdot) \) for a.e. \( x \in \Omega \) shows that (2.2) satisfies the Caratheodory conditions \([9], [11]\). If moreover the condition (2.6) holds then the (Nemytsky) operator \( u \mapsto S(u, \nabla u) \) is a well defined bounded continuous operator from \( L^p(\Omega, \mathbb{R}^m) \) into \( L'^p(\Omega, \mathbb{R}^n) \) \([11]\); in particular this operator is bounded and continuous from \( W^{1,p}(\Omega, \mathbb{R}^m) \) into \( L'^p(\Omega, \mathbb{R}^m) \).

**Remark 2.2** If the mapping (2.2) satisfies all the conditions \((I) \), then

\[
u \mapsto -\text{div} S(u, \nabla u)
\]

(2.7)

is a well defined continuous operator from \( W^{1,p}(\Omega, \mathbb{R}^m) \) into \( W^{-1,p}(\Omega, \mathbb{R}^m) \) (see \([3], [12]\), and taking into account the Green’s formula in Sobolev spaces \([3]\) we obtain

\[
\langle v, -\text{div} S(u, \nabla u) \rangle = \int_{\Omega} S(u, \nabla u) \cdot \nabla v dx, \quad v \in W^{1,p}_0(\Omega, \mathbb{R}^m),
\]

(2.8)

where \( \langle \cdot, \cdot \rangle \) is the pairing duality of \( W^{1,p}(\Omega, \mathbb{R}^m) \) and \( W^{-1,p}(\Omega, \mathbb{R}^m) \).

We note that if \( p > 2 \) then \( W^{1,p}_0(\Omega, \mathbb{R}^m) \subset W^{1,p}(\Omega, \mathbb{R}^m) \) and therefore we have
\[ \langle v, -\text{div} S(u, \nabla u) \rangle = \int_{\Omega} S(u, \nabla u) \cdot \nabla v \, dx, \quad (2.8') \]

for every \((u, v) \in W^{1,p}_0(\Omega, \mathbb{R}^m) \times W^{1,p}(\Omega, \mathbb{R}^m)\).

Consequently, if restrictions (I) hold an \(p > 2\), it results that the operator (2.7) determines in a unique way the bounded and continuous operator \([11]\)

\[
\left\{ \begin{array}{l}
 u \mapsto A(u) \in W^{-1,p}(\Omega, \mathbb{R}^m), \quad u \in W^{1,p}(\Omega, \mathbb{R}^m), \\
 \langle v, A(u) \rangle = \int_{\Omega} S(u, \nabla u) \cdot \nabla v \, dx, \quad v \in W^{1,p'}_0(\Omega, \mathbb{R}^m).
\end{array} \right. \quad (2.9)
\]

(II) Restrictions on \(b(u, \nabla u)\). a) For every \((p, P) \in \mathbb{R}^m \times \mathbb{M}_{\text{mori}}\) the mapping \(b(\cdot, p, P): \Omega \to \mathbb{R}^m\) is measurable, i.e. its real components \(b_i(\cdot, p, P)\) are measurable. b) For a.e. \(x \in \Omega\), the mapping

\[ b(x, \cdot) : \mathbb{R}^m \times \mathbb{M}_{\text{mori}} \to \mathbb{R}^m \]

is Fréchet continuously differentiable. This implies that for a.e. \(x \in \Omega\) there exist the “partial derivatives” of \(b\) with respect to \(p \in \mathbb{R}^m\) and \(P \in \mathbb{M}_{\text{mori}}\)

\[
\left\{ \begin{array}{l}
 (p, P) \mapsto \frac{\partial b}{\partial p}(x, p, P) \in L(\mathbb{R}^m \times \mathbb{R}^m), \\
 (p, P) \mapsto \frac{\partial b}{\partial P}(x, p, P) \in L(\mathbb{M}_{\text{mori}} \times \mathbb{R}^m),
\end{array} \right. \quad (2.10)
\]

which are continuous on \(\mathbb{R}^m \times \mathbb{M}_{\text{mori}}\) and are defined through

\[
\left\{ \begin{array}{l}
 q = (q_i) \mapsto \frac{\partial b}{\partial q}(x, p, P)q := \frac{\partial b}{\partial p_i}(x, p, P)q_i \in \mathbb{R}^m, \quad q \in \mathbb{R}^m, \\
 Q = (Q_{ij}) \mapsto \frac{\partial b}{\partial P}(x, p, P)Q := \frac{\partial b}{\partial P_{ij}}(x, p, P)Q_{ij} \in \mathbb{R}^m, \quad Q \in \mathbb{M}_{\text{mori}}.
\end{array} \right. \quad (2.11)
\]

c) For each \((p, P) \in \mathbb{R}^m \times \mathbb{M}_{\text{mori}}\) the mappings

\[
\frac{\partial b}{\partial p}(\cdot, p, P) : \Omega \to \mathbb{R}^m, \quad \frac{\partial b}{\partial P}(\cdot, p, P) : \Omega \to \mathbb{M}_{\text{mori}}, \quad i = 1, m, j = 1, n,
\]

are measurable. d) The mapping \(b(\cdot, \cdot)\) satisfies the growth condition

\[ |b(x, p, P)| \leq \psi(x) + b^1 |p|^{p-1} + b^2 |P|^{p-1}, \quad \forall (x, p, P) \in \Omega \times \mathbb{R}^m \times \mathbb{M}_{\text{mori}}, \quad (2.12)\]

where \(\psi \in L^p(\Omega)\) and \(b^1 > 0\), \(b^2 > 0\) are constants independent of \((x, p, P)\).

\textbf{Remark 2.3} The condition (II) and the continuity of \(b(x, \cdot, \cdot)\) for a.e. \(x \in \Omega\) shows that the mapping (2.3) satisfies the Carathéodory conditions. If moreover the growth condition (2.12) holds it results that the (Nemytsky) operator

\[ u \mapsto B(u) := b(u, \nabla u) \quad (2.13) \]

is a bounded continuous operator from \(W^{1,p}(\Omega, \mathbb{R}^m)\) into \(L^p(\Omega, \mathbb{R}^m)\).

\textbf{Remark 2.4} In consideration of Remark 2.2 it results that if \(p > 2\) then the operator
\[ u \mapsto -\text{div}\ S(u, \nabla u) - b(u, \nabla u) \]  
\hspace{1cm} (2.14)

from \( W^{1,p}(\Omega, \mathbb{R}^m) \) into \( X(\Omega, \mathbb{R}^m) \) is a continuous operator and in view of (1.3) it follows that, for \( p > 2 \), the equation (2.1) makes sense for \( f \in L^p(\Omega, \mathbb{R}^m) \).

We point out that the operator (2.14) determines in a unique way the bounded and continuous operator

\[
\begin{aligned}
\{&u \mapsto \Lambda(u) := A(u) - B(u) \in X(\Omega, \mathbb{R}^m), \quad u \in W^{1,p}(\Omega, \mathbb{R}^m), p > 2, \\
&\{v, \Lambda(u)\} = \int_\Omega [S(u, \nabla u) \cdot \nabla v - b(u, \nabla u) \cdot v] \, dx, \quad v \in W_0^{1,p}(\Omega, \mathbb{R}^m). \}
\end{aligned}
\]

(2.15)

3. WEAK SOLUTIONS OF THE DIRICHLET PROBLEM FOR THE SYSTEM (2.1)

In the conditions of the preceding Section we have in view to prove the existence of weak solutions \( u \in W^{1,p}(\Omega, \mathbb{R}^m), p > 2 \), of the Dirichlet problem

\[
\begin{cases}
-\text{div}\ S(u, \nabla u) - b(u, \nabla u) = f & \text{in } \Omega, \\
u = u_0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( f \in L^p(\Omega, \mathbb{R}^m) \) and \( u_0 \in W^{1,p}(\partial \Omega, \mathbb{R}^m) \).

The function \( u \in W^{1,p}(\Omega, \mathbb{R}^m) \) is called a weak solution to the problem (P) if \( u \) is the solution of the variational problem

\[
\begin{aligned}
\{&\Lambda(u) = f, \quad u - g \in W_0^{1,p}(\Omega, \mathbb{R}^m), \\
&\Lambda(\Lambda^\circ u) = \Lambda(g + u) = f, \quad u \in W_0^{1,p}(\Omega, \mathbb{R}^m), \\
&\{v, \Lambda^\circ(u) - f\} = \int_\Omega [\nabla v \cdot S(g + u, \nabla(g + u)) - v \cdot [b(g + u, \nabla(g + u)) - f]] \, dx = 0, \\
\}
\end{aligned}
\]

(3.1)

(3.1’)

for every \( v \in W^{1,p}(\Omega, \mathbb{R}^m) \).

Further we are going to use the following

\textbf{Lemma 3.1} For every \( g \in W^{1,p}(\Omega, \mathbb{R}^m) \) and \( u, v \in W^{1,p}(\Omega, \mathbb{R}^m) \) we have

\[
\{u - v, \Lambda^\circ(u) - \Lambda^\circ(v)\} = L_0(g, u, v) + L_1(g, u, v),
\]

(3.2)

Where

\[
\begin{aligned}
L_0(g, u, v) &= \int_0^1 \int_\Omega \left\{ \frac{\partial S}{\partial p}(g + w, \nabla(g + w)) h + \frac{\partial S}{\partial p}(g + w, \nabla(g + w)) \nabla h \right\} \, dx, \\
L_1(g, u, v) &= \int_0^1 \int_\Omega \left\{ \frac{\partial b}{\partial p}(g + w, \nabla(g + w)) h + \frac{\partial b}{\partial p}(g + w, \nabla(g + w)) \nabla h \right\} \, dx,
\end{aligned}
\]

(3.3)

and \( w = v + th, \quad h = u - v \).

\textbf{PROOF:} From (2.15) we obtain
\[ \langle u - v, \Lambda^\varepsilon(u) - \Lambda^\varepsilon(v) \rangle = \]
\[ = \int_\Omega \nabla h \cdot [S(g + u, \nabla (g + u)) - S(g + v, \nabla (g + v))] \, dx - \int_\Omega h \cdot [b(g + u, \nabla (g + u)) - b(g + v, \nabla (g + v))] \, dx = \]
\[ = \int_\Omega [\nabla h \cdot \frac{dS}{dt}(g + w, \nabla (g + w)) \, dr] \, dx - \int_\Omega [h \cdot \frac{db}{dt}(g + w, \nabla (g + w)) \, dr] \, dx \]

where \( w = v + th \), \( h = u - v \). Taking into account that \( S(x, \cdot, \cdot) \) and \( b(x, \cdot, \cdot) \) are Fréchet differentiable and applying the Chain Rule we get (3.2).

4. AN EXISTENCE RESULT OF THE PROBLEM (P)

**THEOREM 4.1** If for every \( u, h \in W^{1,p}(\Omega, \mathbb{R}^m) \) and \( (x, p, P) \in \Omega \times \mathbb{R}^n \times M_{\text{meas}} \) we have

\[
(H_1) \quad \begin{cases} 
\int_0^1 \int_\Omega \nabla h \cdot \frac{dS}{dp}(u + th, \nabla (u + th)) \nabla h \, dx \geq c_0 \| h \|^p_{L^p}, \\
\int_0^1 \int_\Omega \nabla h \cdot \frac{db}{dp}(u + th, \nabla (u + th)) h \, dx \geq 0 
\end{cases}
\]

and

\[
(H_2) \quad \begin{cases} 
\int_0^1 \int_\Omega h \cdot \frac{db}{dp}(u + th, \nabla (u + th)) h \, dx \leq 0 \\
\frac{\partial b}{\partial p}(x, p, P) \leq c_1 (1 + |p|^{p-1} + |P|^{p-1}), 
\end{cases}
\]

where \( q \in (1, p - 1) = (1, p/p') \), and \( c_0 > 0 \), \( c_1 > 0 \) are constants independent of \( u \), \( h \) and \( (x, p, P) \), then the bounded and continuous operator

\[
\begin{cases} 
\{ u \mapsto \Lambda^\varepsilon(u) \in W^{1,p}(\Omega, \mathbb{R}^m), & u \in W_0^{1,p}(\Omega, \mathbb{R}^m), \\
\{ v, \Lambda^\varepsilon(u) \} = \int_\Omega [\nabla v \cdot S(g + u, \nabla (g + u)) - v \cdot b(g + u, \nabla (g + u))] \, dx, & v \in W^{1,p}_{L^p}(\Omega, \mathbb{R}^m), 
\end{cases}
\]

(4.1)
is a Gårding coercive operator.

**PROOF:** A. **The operator (4.1) is a Gårding operator.** In view of imbeddings (1.1) and (1.2) we can chose \( V = W_0^{1,p}(\Omega, \mathbb{R}^m) \) and \( U = L^p(\Omega, \mathbb{R}^m) \) in Def. 1.1 of Gårding operators. In this definition we take [10]

\[
(u, v) \mapsto F(u, v) := \Lambda^\varepsilon(u) + 0(v) = \Lambda^\varepsilon(u) \in W^{-1,p'}(\Omega, \mathbb{R}^m), \quad u, v \in W_0^{1,p}(\Omega, \mathbb{R}^m),
\]

(4.2)

where \( 0 \) is the null operator. With this choice, the condition \((iii)\) in Def. 1.1 is trivially satisfied because \( F(v, u_0) - F(v, u) = 0 \) for every \( v \in W_0^{1,p}(\Omega, \mathbb{R}^m) \). The condition \((i)\) of Def. 1.1 is fulfilled since \( F(u + tv, w) = \Lambda^\varepsilon(u + tv) \) for every \( u, v, w \in W_0^{1,p}(\Omega, \mathbb{R}^m) \) and \( t \in \mathbb{R} \), and the real function

\[
t \mapsto \{ v, \Lambda^\varepsilon(u + tv) \} = \int_\Omega [\nabla v \cdot S(u + tv, \nabla (u + tv)) - v \cdot b(u + tv, \nabla (u + tv))] \, dx \in \mathbb{R}, t \in \mathbb{R}
\]
is continuous in consideration of condition \((\text{I})_b\) and \((\text{II})_b\). Consequently, to prove that \(\Lambda^g\) is a Gårding operator we have only to show that, with \(F\) given by (4.2), the condition \((ii)\) of Def. 1.1 is verified.

Taking into account \((\text{II}_1)\) and \((\text{II}_2)\) we get

\[
L_0(g,u,v) \geq c_0 \| u - v \|_{p,p}^p , \tag{4.3}
\]

\[
-L_1(g,u,v) \leq c_1 \int_\Omega | h | \| \nabla h \| \, dx \left( [1 + \| g + w \|_{p+1} + | \nabla g + \nabla w \|_{p+1}] \right) dt \leq \]

\[
\leq c_1 \int_0^1 dt \int_\Omega | h | \| \nabla h \| \left( [1 + 2^{p-1}(\| g \|_{p+1} + | \nabla g \|_{p+1} + (\| v + th \|_{p+1} + | \nabla (v + th) \|_{p+1})] \right) dx. \tag{4.4}
\]

Using some elementary results from the theory of \(L^p(\Omega)\) spaces ([1], [2], [7]), taking into account that \(\Omega \subset \mathbb{R}^n\) is a bounded Lipschitz domain we obtain: 

\(a)\) Because \(h \in W^{1,p}(\Omega, \mathbb{R}^m)\) it follows that \(\| h \|, \| \nabla h \| \in L^p(\Omega) \subset L^2(\Omega)\) and therefore

\[
\int_\Omega | h | \| \nabla h \| \, dx \leq \| h \|_p \| \nabla h \|_p \leq \text{const.} \| h \|_p \| \nabla h \|_p
\]

(4.5)

since \(\| \cdot \|_p \leq \text{const.} \| \cdot \|_p\). 

\(b)\) Let us point out the implications:

\[
q \in (1, p-1) \Rightarrow 0 < p(q-1)/(p-2) \Rightarrow \frac{p(q-1)}{p-2} < p \Rightarrow L^p(\Omega) \subset L^q(\Omega) , \quad s = p(q-1)/(p-2),
\]

\(g \in W^{1,p}(\Omega, \mathbb{R}^m) \Rightarrow \| g \| \in L^p(\Omega) \subset L^q(\Omega) \Rightarrow \| g \|_{p-2} \in L^q(\Omega)\).

As \(p^{-1} + p^{-1} + (\frac{p}{p-2})^{-1} = 1\), by virtue of generalized Hölder inequality ([2], [7]), it results that

\[
\| h \| \| \nabla h \| \| g \|_{p-2} \in L^1(\Omega)
\]

and

\[
\int_\Omega | h | \| \nabla h \| \| g \|_{p-2} \, dx \leq \| h \|_p \| \nabla h \|_p \| g \|_{p-2} \|_{p-2}.
\]

On the other hand we have \(\| g \|_{p-2} \|_{p-2} = \| g \|_{p-2} \| g \|_{p-2}\), whereof we obtain

\[
\int_\Omega | h | \| \nabla h \| \| g \|_{p-2} \, dx \leq \| h \|_p \| \nabla h \|_p \| g \|_{p-2} \| \leq \text{const.} \| h \|_p \| \nabla h \|_p.
\]

(4.6)

\(c)\) From the implications \(g \in W^{1,p}(\Omega, \mathbb{R}^m) \Rightarrow \| \nabla g \| \in L^p(\Omega) \subset L^q(\Omega) \Rightarrow \| \nabla g \|_{p-2} \in L^q(\Omega)\), \(h \in W^{1,p}(\Omega, \mathbb{R}^m) \Rightarrow \| h \| \| \nabla h \| \in L^p(\Omega)\), and from \(p^{-1} + p^{-2} + (\frac{p}{p-2})^{-1} = 1\) it results that \(\| h \| \| \nabla h \| \| \nabla g \|_{p-2} \in L^1(\Omega)\) and

\[
\int_\Omega | h | \| \nabla h \| \| \nabla g \|_{p-2} \, dx \leq \| h \|_p \| \nabla h \|_p \| \nabla g \|_{p-2} \|_{p-2}.
\]

As \(\| \nabla g \|_{p-2} \|_{p-2} = \| \nabla g \|_{p-2} \| g \|_{p-2}\) it follows

\[
\int_\Omega | h | \| \nabla h \| \| \nabla g \|_{p-2} \, dx \leq \| h \|_p \| \nabla h \|_p \| \nabla g \|_p \| \leq \text{const.} \| h \|_p \| \nabla h \|_p.
\]

(4.7)
Similarly with (4.6) and (4.7) we obtain
\[ \int_{\Omega} |h||\nabla h||v+th|^{q-1} \, dx \leq \text{const.} ||h||_p ||h||_{1,p} ||v+th||_p^{q-1}, \] (4.8)
\[ \int_{\Omega} |h||\nabla \nabla (v+th)|^{q-1} \, dx \leq \text{const.} ||h||_p ||h||_{1,p} ||\nabla (v+th)||_p^{q-1}. \] (4.9)

From (4.4) and (4.5)–(4.9) we have
\[ -L_1(g,u,v) \leq \text{const.} ||h||_p ||h||_{1,p} (1+||v+\theta h||_p^{q-1} + ||\nabla (v+\theta h)||_p^{q-1}), \] (4.10)
and, on the other hand
\[ \begin{align*}
&\left\|v+\theta h\right\|_p \leq \left\|v+\theta h\right\|_{1,p} + 2
\|v\|_{1,p}, \\
&\left\|\nabla (v+\theta h)\right\|_p \leq \left\|v+\theta h\right\|_{1,p} + 2
\|v\|_{1,p},
\end{align*} \] \( \theta \in (0,1). \) (4.11)

In view of the dense imbedding (1.2) and (4.11) we have
\[ \|v+\theta h\|_p^{q-1} \leq \text{const.} r^{q-1}, \quad \|\nabla (v+\theta h)\|_p^{q-1} \leq \text{const.} r^{q-1}, \] (4.12)
for every \( u,v \in B_r(0) = \{u \in W_0^{1,p}(\Omega,\mathbb{R}^m) \mid ||u||_p < r\} \). From (4.10) and (4.12) it results
\[ -L_1(g,u,v) \leq \text{const.} ||h||_p ||h||_{1,p} (a_1 + a_2 r^{q-1}), \quad h = u - v, \quad u,v \in B_r(0), \] (4.13)
where \( a_1 > 0 \) and \( a_2 > 0 \) are constants depending on \( \Omega, p \), and \( g \). By using a variant of the Young inequality [2] we get
\[ \begin{align*}
&\left\|h\right\|_p \left\|h\right\|_{1,p} \leq \varepsilon \left\|h\right\|_p^p + c(\varepsilon) \left\|h\right\|_p^{p'}, \\
&\left\|h\right\|_p \left\|h\right\|_{1,p} r^{q-1} \leq \varepsilon \left\|h\right\|_p^p + c(\varepsilon) \left\|h\right\|_p^{p'} r^{(q-1)p'},
\end{align*} \] (4.14)
where \( \varepsilon > 0 \) is an arbitrary constant, \( c(\varepsilon) = \varepsilon^{1/(p-1)} \), and \( h = u - v \). Therefore, from (4.13), (4.14) we have
\[ -L_1(g,u,v) \leq (a_1 + a_2) \varepsilon \left\|h\right\|_p^p + (a_1 + a_2 r^{q-1} \varepsilon) c(\varepsilon) \left\|h\right\|_p^{p'}, \] (4.15)
whereof, in view of (4.3), from (3.2) it results
\[ \left\langle u - v, \Lambda^\varepsilon(u) - \Lambda^\varepsilon(v) \right\rangle \geq c_0 \left\|h\right\|_p^p -(a_1 + a_2) \varepsilon \left\|h\right\|_p^p -(a_1 + a_2 r^{q-1} \varepsilon) c(\varepsilon) \left\|h\right\|_p^{p'} . \] (4.16)

If in this inequality we take \( \varepsilon > 0 \) sufficiently small, it follows that for every \( u, v \in B_r(0) \) we have
\[ \left\langle u - v, \Lambda^\varepsilon(u) - \Lambda^\varepsilon(v) \right\rangle \geq b_0 \left\|u - v\right\|_p -(b_1 + b_2 r^{(q-1)p'}) \left\|u - v\right\|_p^{p'}, \] (4.17)
where \( b_0 > 0, b_1 > 0, \) and \( b_2 > 0 \) are constant. Thus we proved that \( \Lambda^\varepsilon \) is a Gårding operator for every \( g \in W^{1,p}(\Omega,\mathbb{R}^m) \) since (4.16) implies
\[ \left\langle u - v, \Lambda^\varepsilon(u) - \Lambda^\varepsilon(v) \right\rangle \geq -\gamma(r, \left\|u - v\right\|_p), \quad u,v \in B_r(0), \] (4.17)
where \( \gamma(x,y) = (b_1 + b_2 x^{(q-1)p'}) y^{p'}, \ x \geq 0, \ y \geq 0, \) satisfies \( \lim_{\delta \downarrow 0} \theta^{-1} \gamma(x,\theta y) = 0, \ \forall x, \ y > 0. \)

**B. The operator (4.1) is coercive.** By taking \( v = 0 \) in (3.2) we obtain
\[
\{ u, \Lambda^\varepsilon(u) \} = L_0(g, u, 0) + L_1(g, u, 0) + \{ u, \Lambda^\varepsilon(0) \},
\] (4.18)

From (3.3) with \( v = 0 \) and hypotheses \((H_1)\), \((H_2)\) we obtain
\[
\begin{aligned}
L_0(g, u, 0) &\geq c_0 \| u \|_{1,p}^p, \\
-L_1(g, u, 0) &\leq \frac{1}{\varepsilon} \int_0^1 \| u \| \| \nabla u \| \{ 1 + 2^{q-1} \| g \|_{q^{-1}} + \| \nabla g \|_{q^{-1}} + \varepsilon \| u \|_{q^{-1}} + \| \nabla u \|_{q^{-1}} \} \, dx.
\end{aligned}
\] (4.19)

If in estimations (4.5)–(4.9) we take \( v = 0 \) and use the Young inequality as in (4.4) we obtain
\[
-L_1(g, u, 0) \leq A_1 \| u \|_{1,p} + A_2 c(\varepsilon) \| u \|_{1,p} + A_3 c(\varepsilon) \| u \|_{1,p},
\] (4.20)

where \( A_i > 0, A_2 > 0, A_3 > 0 \) are constants depending on \( \Omega, q, g \), and \( \varepsilon > 0 \) is an arbitrary constant.

Applying successively H"older and Young inequalities we have
\[
\begin{aligned}
&\| u, \Lambda^\varepsilon(0) \|_{p'} \leq \| S(g, \nabla g) \|_{p'} + \| b(g, \nabla g) \|_{p'} + \| u \|_{1,p} \leq \\
&\| S(g, \nabla g) \|_{p'}^p + \| b(g, \nabla g) \|_{p'}^p + c(\varepsilon) \| u \|_{p'} + \| \nabla u \|_{p'} \leq \\
&\leq B_1 + B_2 \| u \|_{1,p},
\end{aligned}
\] (4.21)

where \( B_1 > 0 \) and \( B_2 > 0 \) are constants with evident dependence on \( S \) and \( b \). From (4.18)–(4.21) we obtain
\[
\begin{aligned}
\{ u, \Lambda^\varepsilon(u) \} &\geq (c_0 - A_1 \varepsilon) \| u \|_{p} - (A_2 c(\varepsilon) + B_2) \| u \|_{1,p} - A_3 c(\varepsilon) \| u \|_{1,p} - B_1, \\
\end{aligned}
\] (4.22)

where \( \varepsilon > 0 \) is an arbitrary constant. If we take \( \varepsilon > 0 \) sufficiently small in (4.22) it results
\[
\begin{aligned}
\{ u, \Lambda^\varepsilon(u) \} &\geq C_0 \| u \|_{1,p} - C_1 \| u \|_{1,p} - C_2 \| u \|_{1,p} - B_1, \\
\end{aligned}
\] (4.23)

from where we get
\[
\| u \|_{1,p}^{-1} \{ u, \Lambda^\varepsilon(u) \} \geq \| u \|_{1,p}^{-1} \{ C_0 - C_1 \| u \|_{1,p} - C_2 \| u \|_{1,p} - B_1 \| u \|_{1,p} \} \\
\] (4.24)

for every \( u \in W_0^{1,p}(\Omega, \mathbb{R}^m) \). Since \( p - 1 > 1 \), \( p' - p < 0 \), and \( q p' < p \) we obtain
\[
\| u \|_{1,p}^{-1} \{ u, \Lambda^\varepsilon(u) \} \rightarrow \infty \text{ as } \| u \|_{1,p} \rightarrow \infty, \quad u \in W_0^{1,p}(\Omega, \mathbb{R}^m),
\]
and the theorem is proved (see (1.4)).

**Remark 4.1** Because a bounded Gårding operator is pseudomonotone [11], it follows the implication [8]
\[
\begin{aligned}
\text{If } u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega, \mathbb{R}^m) \text{ and} \\
\limsup_{n \rightarrow \infty} \{ u_n - u, \Lambda^\varepsilon(u_n) \} \leq 0
\end{aligned}
\]
\[
\Rightarrow \liminf_{n \rightarrow \infty} \{ u_n - v, \Lambda^\varepsilon(u_n) \} \geq \{ u - v, \Lambda^\varepsilon(u) \},
\]

for every \( v \in W_0^{1,p}(\Omega, \mathbb{R}^m) \).

From theorems 1.1 and 4.1 we obtain the desired existence result.
THEOREM 4.2 If $\Omega \subset \mathbb{R}^n$ is a Lipschitz bounded domain, $p > 2$, and the mappings (2.2) and (2.3) satisfy the restrictions (I) and (II) of Section 2 and the hypotheses $(H_1)$ and $(H_2)$ in theorem 4.1 then, for every pair $(f, u_0) \in L^p(\Omega, \mathbb{R}^m) \times W^{1,p}_0(\Omega, \mathbb{R}^m)$, there exists at least one weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ of the problem (P).

Remark 4.2 From the proof of lemma 3.1 and hypothesis $(H_1)$, it results that the operator 
$$ u \mapsto A^g(u) := A(g + u) \in W^{-1,p}(\Omega, \mathbb{R}^m), \quad u \in W^{1,p}(\Omega, \mathbb{R}^m), $$
defined by (2.9), is a $p$-coercive, and consequently a strongly monotone operator ([9]) for every $g \in W^{1,p}(\Omega, \mathbb{R}^m)$, i.e.
$$ \langle u - v, A^g(u) - A^g(v) \rangle \geq c_0 \|u - v\|_{-1,p}^p, \quad c_0 > 0, \forall u, v \in W^{1,p}(\Omega, \mathbb{R}^m). $$

Remark 4.3 If the mapping (2.2) is independent of $u$ and $m = n \geq 1$ then the system (2.1) is a quasilinear differential system of finite $n$-dimensional elastostatics type. In [4] we obtained some existence results of the weak solutions to the Dirichlet problem for such a system in three dimensions.

REFERENCES


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