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ON NESTED ATTRACTORS OF COMPLEX CONTINUOUS SYSTEMS DETERMINATION

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This paper deals with the determination of attractors and attraction zones of continuous non linear systems, based on aggregation technics for stability study and the choice of their state representations for description. Aggregation technique by the use of vectors norms enable to determine comparison systems of a complex non linear and/or non stationary model of a given process. The study of the stability of this comparison system enables to easily study the stability of the process and to define attraction domains. When the system is locally unstable, the implementation of comparison systems permits to determine an attractor of the studied system in a given attraction domain. This work presents an approach which enables to check the quality of control law for a process whose evolution is described by the equation state, and the control is made from accessible information about the process and its desired evolution.

Key words: stability, overvaluing systems, attractor, attraction domain.

1. INTRODUCTION

The problem of stability analysis of non linear systems has received considerable attention in the field of research in automatic control and different approaches have been proposed in the literature related with this subject [1, 3, 4].

The stability theory aims at drawing conclusions about the behaviour of a system without actually computing its solution trajectories. Lagrange (1788) concluded that, in the absence of external forces, an equilibrium state of a conservative mechanical system is stable provided that it corresponds to a minimum of the potential energy. Then, for a long period, stability studies have been limited to conservative mechanical systems described by Lagrangian equations of motion. The quantum advance in stability theory that allowed one the analysis of arbitrary differential equations is due to A. M. Lyapunov (1892).He introduced the basic idea and the definitions of stability that are in use today and proved many of the existing fundamental theorems [16]. The concept of Lyapunov stability plays an important role in control and system theory. Solutions of stability problems risen from large scale systems, have been approached [7, 8, 15, 17] by Bellman's concept (1962) of vector Lyapunov functions and Bailey and Siljak approaches to find conditions and interactions under which stability property of the overall system is inferred from stability properties of subsystems, and Robert's concept (1964) of vector norms, and its various applications by Borne et al (1973), Borne and Gentina (1974) and Benrejeb (1982).

For large scale systems, generally described in state space, stability conditions are obtained, for the whole system or for subsystems and interactions [9, 11, 14, 18]. The influence of the state vector description and of the matrix characteristic, on the determination of the stability domain, is based on Borne-Gentina stability criteria [6, 9, 11, 12, 13], with the use of vector norms and of overvaluing systems of nonlinear large scale systems [2, 4, 5].

The aim of this work is to define a procedure enable to determine attraction domains corresponding to adequate representations of the non linear studied systems. In section 2, is presented the determination of a comparison system and in section 3 the determination of attractors. The determination of nested attractors for a second order complex continuous system is proposed in section 4, to illustrate the efficiency of the proposed approach.

2. DETERMINATION OF COMPARISON SYSTEMS

For stability study, several criteria based on different Lyapunov's theorems are developed. If applied to the comparison system, enable to determine the attractor and its field of attraction [17].

Let us consider the following system (S) described by

$$\text{(S)}: \dot{x} = A(x,t)x + B(x,t), \qquad \forall\, t \in \tau_0 = [t_0, +\infty]. \tag{1}$$

Let p be a regular Vector Norm (VN) of size k and S a compact set of $\Re^n$ which includes the origin.

***Definition* 1.** The pair $(M,\ N), M: \tau_0 \times S \to \Re^{k\times k}, N: \tau_0 \times S \to \Re_+^k$ defines a non homogeneous pseudo-Overvaluing System (NHOS) of system (S) on the compact set S closed to the (VN) p (annex), if and only if we have

$$D^+ p(x) \le M(t,x)\,p(x) + N(t,x), \forall (t,x) \in \tau_0 \times S. \tag{2}$$

$D^+ p(x)$ is the right-hand derivative, taken along the motion of (S) into E, and $\tau_0 = [t_0, +\infty]$.

***Definition* 2.** [11]. The real parts of the eigenvalues of matrix A, with non negative off diagonal elements, are less than a real number μ if and only if all those of matrix M, $M = \mu I_n - A$, are positive, with I_n the n identity matrix.

When successive principal minors of matrix $(-A)$ are positive, Kotelyanski lemma permits to conclude on stability property of the system characterized by A.

THEOREM 3. [6]. *If it is possible to define, in a domain D, a time-invariant linear overvaluing system of (1) related to a regular vector norm* p

$$D^+ p(x) = Mp(x) + N, \forall (x) \in \tau_0 \times S, \tag{3}$$

for which M is the opposite of a (constant) M-matrix, and N is a non negative (constant) vector, then, there is an asymptotically stable attractor L_0 *and the set*

$$L_0 = \left\{ x \in \mathbb{R}^n ; p(x) \le -M^{-1}N \right\} \tag{4}$$

includes all the attractors of (1), if the outer frontier of D encloses, also, the outer frontier of L_0.

3. PROPOSED ATTRACTORS DETERMINATION

Let us consider the non linear continuous complex systems described in state space by

$$\dot{x} = f(x,t), \tag{5}$$

where $x(t)$ is the state space vector at time t, $x \in \Re^n$, and $f: \Re^n \times \Re_+ \to \Re^n$ a non linear vector function.

Consider, as a first step, the domain D_1 of x, for which the system (5) can be described in the form

$$\dot{x} = A(x,t)x + B(x,t), \tag{6}$$

where A is an $n \times n$ matrix and B an n vector.

The use of aggregation technique and particularly of the (VN) *p(x)* (annex 1), in D_1, is supposed to lead to a linear comparison system of the form

$$\dot{z} = Mz + N, \tag{7}$$

with

$$p(x) \le z \tag{8}$$

Then, if the overvaluing matrix M of matrix A is the opposite of an M-matrix [9–11], it comes

$$\lim_{t\to\infty} p(x) \le z_{\max} \quad \text{if} \quad p(x_0) \le z_0 \tag{9}$$

and $z_{\max}$ is equal to the solution

$$z_{\max} = -M^{-1}N \tag{10}$$

of the following equation

$$M z_{\max} + N = 0.$$

This constraints on z limit the asymptotic evolution of the state space vector x in a new domain $D_2 \subset D_1$, defined by (9) and (10).

As a second step, it can be useful to look for a new description of the system (1) for $x \in D_2$.

Then, it is sufficient to repeat this approach to obtain a new attractor D_3 of *x,* and then smallest other domains can be determined by iteration of the proposed procedure.

If the attractor is reduced to the point zero (0) the system is stable in the usual definition and zero is the stable equilibrium point [9-11].

The used comparison systems depend of the choice of the vector norm that can influence the efficiency of the proposed attractors determination procedure, as shown in the next section.

4. APPLICATION TO A NON LINEAR COMPLEX SYSTEM

To illustrate the proposed approach for attractors determination, let us consider the system (S) defined by (1), with

$$A\big(x(t)\big) = \begin{bmatrix} -3 - x_1^2 & 0.3 + \text{sat}x_2 \\ \sin x_1 x_2 & -1 + 0.1\cos x_1 \end{bmatrix}, \tag{11}$$

$$B\big(x(t)\big) = \begin{bmatrix} -\,\text{sat}x_1 \\ 0.1 + 0.2\sin x_1 \end{bmatrix} \tag{12}$$

and

$$\text{sat}v = v \ \text{ if } \ |v| \le 1, \tag{13}$$

$$\text{sat}\, v = \text{sig}\, v \ \text{ if } \ |v| \succ 1. \tag{14}$$

The convergence of the trajectories towards the equilibrium $x = 0$ will be studied through the convergence of the vector norm

$$p(x) = \big[|x_1|, |x_2|\big]^T .$$

As a first step, the system (S) is supposed to lead, in a domain D_1, to a comparison system in the form (7).

Step1: Detrenination of the attractor D_2

If the comparison system of the process is on the form (7), according to (S), the minimal overvaluing matrix relatively to the regular vector norm $p(x)$ is the following

$$M\big(A\big(x(t)\big)\big) = \begin{bmatrix} -3 - x_1^2 & |0.3 + \text{sat}x_2| \\ |\sin x_1 x_2| & -1 + 0.1\cos x_1 \end{bmatrix} \tag{15}$$

and

$$N(B(x(t)))=\begin{bmatrix}|-\text{sat}x_1|\\|0.1+0.2\sin x_1|\end{bmatrix}. \tag{16}$$

In this case, the comparison system is described by

$$\dot{z}=\begin{bmatrix}-3 & 1.3\\1 & -0.9\end{bmatrix}z+\begin{bmatrix}1\\0.3\end{bmatrix}. \tag{17}$$

By applying the Borne and Gentina Stability conditions (annex 2)

$$\begin{cases}-(-3)\succ 0\\(-1)^2\det M\succ 0\end{cases}.$$

It comes from (9) and (10)

$$z_\infty=\begin{bmatrix}0.9214\\1.3571\end{bmatrix}\geq\begin{bmatrix}|x_1|\\|x_2|\end{bmatrix}.$$

Hence, the corresponding attractor zone $D_2\subset D_1$ is defined by

$$D_2=\{x,|x_1|<0.9214\ ,|x_2|<1.3571\}. \tag{18}$$

Step 2: Determination of the attractor D_3

A new description of the system (S) can be defined, in D_2, by a new description in the form (6). Indeed, $|x_1|\prec 0.921$ so $\text{sat}\,x_1=x_1$

Hence the description

$$A(x(t))=\begin{bmatrix}-4-x_1^2 & 0.3+\text{sat}x_2\\\sin x_1x_2 & -1+0.1\cos x_1\end{bmatrix} \tag{19}$$

and

$$B(x(t))=\begin{bmatrix}0\\0.1+0.2\sin x_1\end{bmatrix}, \tag{20}$$

then the new attractor characterized by

$$M(A(x(t)))=\begin{bmatrix}-4-x_1^2 & |0.3+\text{sat}x_2|\\|\sin x_1x_2| & -1+0.1\cos x_1\end{bmatrix}, \tag{21}$$

$$N(B(x(t)))=\begin{bmatrix}0\\|0.1+0.2\sin x_1|\end{bmatrix}, \tag{22}$$

with

$$|x|\leq\begin{bmatrix}0.9214\\1.3571\end{bmatrix}.$$

In this case, one can easily obtain the linear and stable comparison system in the form (7) such that

$$\dot{z}=\begin{bmatrix}-4 & 1.3\\0.95 & -0.9\end{bmatrix}z+\begin{bmatrix}0\\0.260\end{bmatrix}. \tag{23}$$

It comes from (9) and (10)

$$\lim_{t\to\infty} p(z) \le z_\infty = -M^{-1}N$$

$$z_\infty = \begin{bmatrix} 0.1429 \\ 0.4397 \end{bmatrix} \ge \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}$$

Hence, the corresponding attractor zone $D_3 \subset D_2$

$$D_3 = \{x, |x_1| < 0.1429 \,, |x_2| < 0.4397\} \tag{24}$$

In this domain D_3, we repeat the overvaluation approach to determine a new attractor taking into account the condition

$$|x| \le \begin{bmatrix} 0.1429 \\ 0.4397 \end{bmatrix}$$

Step 3: Determination of the attractor D_4

The domain D_3 is defined such that

$$|x| \le \begin{bmatrix} 0.1429 \\ 0.4397 \end{bmatrix}$$

and $\operatorname{sat} x_2 = x_2 \;\; |x_2| \le 0.4397$.

If the correspondant comparison system, in the form (7)

$$\dot{z} = \begin{bmatrix} -4 & 0.738 \\ 0.063 & -0.9 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0.130 \end{bmatrix} \tag{25}$$

is such that M is the opposite of an M-matrix, we obtain (9), then

$$z_\infty = \begin{bmatrix} 0.027 \\ 0.146 \end{bmatrix} \ge \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}.$$

It comes, the attractor zone D_4

$$D_4 = \{x, |x_1| < 0.027 \,, |x_2| < 0.146\}. \tag{26}$$

Attraction zones and state space variables evolutions of the systems in the nested domains D_1, D_2 and D_3 are given in Fig. 1.

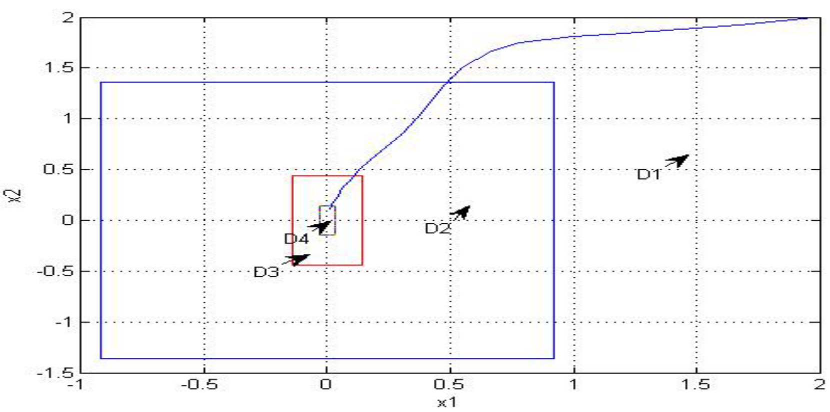

Fig. 1 – Attraction zones and state space variables evolutions of the systems in the domains D_1, D_2, D_3 and D_4.

5. CONCLUSION

The concept of vector norm, associated to the definition of comparison systems defined by Borne and Gentina stability approach are used, in this paper, with success, to determine different and nested attractors. For stable systems, the attractor domain can be reduced to zero. The studied illustrative examples show the efficiency of the proposed procedure.

This method enables to test the accuracy of a controlled system by providing an overvaluation for the error.

ANNEX 1: VECTOR NORMS DEFINITION

***Definition* 1.** Let $E=\Re^n$ and $E_1, E_2 \ldots E_k$ subspaces of the space $E, E=E_1\cup E_2\ldots\cup E_k$ Let x be an n vector defined on E and $x_i=P_ix$ the projection of x on E_i, where P_i is a projection operator from E into E_i, p_i a scalar norm (i =1,2,…, k) defined on the subspace E_i and p denote a Vector Norm (VN) of dimension k and with its component

$$p_i(x)=p_i(x_i)\quad,\qquad p(x):\Re^n\to\Re_+^k.$$

Let y be another vector in space E, with $y_i=P_iy$, we have

$$\begin{cases} p_i(x_i)\geq 0, \forall x_i\in \mathrm{E_i} \forall i=1,2,\ldots,k\\ p_i(x_i)=0\leftrightarrow x_i=0, \forall i=1,2,\ldots,k\\ p_i(x_i+y_i)\leq p_i(x_i)+p_i(y_i), \forall x_i,y_i\in \mathrm{E_i}\ \forall i=1,2,\ldots,k\\ p_i(\lambda x_i)=|\lambda|\,p_i(x_i), \forall x_i\ \forall i=1,2,\ldots,k.\end{cases}$$

If k–1 of the subspaces E_i are insufficient to define the whole space E, the VN is surjective. If in addition the subspaces E_i are in disjoint pairs, $E_i\cap E_j=\varnothing$, $\forall i\neq j=1,2,\ldots,k$, the VN p is said to be regular.

ANNEX 2: BORNE AND GENTINA PRATICAL STABILITY CRITERION [9-11]

Let consider the nonlinear continuous process described in state space by: $\dot{x}=Ax$; A is an $n\times n$ matrix, $A=\{a_{i,j}\}$. If the overvaluing matrix $M(A)$ has its non constant elements isolated in only one row, the verification of the Kotelyanski condition enables to conclude to the stability of the initial system.

As an example, if the non constant elements are isolated in only one row of A, Kotelyanski lemma applied to the overvaluing matrix obtained by the use of the n regular vector norm $p(x)$ with $x=[x_1,x_2,\ldots,x_n]^T$, such that $p(x)=[|x_1|,|x_2|,\ldots,|x_n|]^T$, leads to the following stability conditions of initial system

$$a_{1,1}\prec 0, \begin{vmatrix} a_{1,1} & |a_{1,2}| \\ |a_{2,1}| & a_{2,2}\end{vmatrix}\succ 0,\ldots,(-1)^n\begin{vmatrix} a_{1,1} & |a_{1,2}| & \cdots & |a_{1,n}| \\ |a_{2,1}| & a_{2,2} & \cdots & |a_{2,n}| \\ \vdots & \vdots & & \vdots \\ |a_{n,1}| & |a_{n,2}| & \cdots & |a_{n,n}|\end{vmatrix}\succ 0.$$

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