EQUILIBRIUM PROBLEMS OVER PRODUCT SETS

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Some types of equilibrium problems and systems of equilibrium problems on cones are studied. For these, we obtain equivalence results and prove, using a fixed-point theorem of Chowdury and Tan, existence results.

Key words: equilibrium problems, product sets, hemicontinuity, convexity, generalized pseudomonotonicity.

1. INTRODUCTION

The equilibrium problems were introduced in [7] and [11]. After that, systems of equilibrium problems were first considered in [3]. Since then, many different classes of such systems were studied [1, 2, 4, 5]. Here, we present some new classes of equilibrium problem systems over product set. These results represent, in a certain sense, a generalization of those obtained in [6] for variational inequalities over product sets.

2. FORMULATION OF EQUILIBRIUM MODELS AND SOME PRELIMINARY RESULTS

Let $I = \{1, 2, \ldots, m\}$ be a finite index set and $X_i$, for each $i \in I$, be a real topological vector space, with $K_i$ a nonempty convex subset. We put $X = \prod_{i \in I} X_i$ and $K = \prod_{i \in I} K_i$. For $x_i \in X_i$, $i \in I$, we denote $x = (x_i)_{i \in I} \in X$. For a real topological vector space $Y$, let $C$ be a proper, closed and convex cone with $\text{int} \ C \neq \emptyset$, where $\text{int} \ C$ denotes the topological interior of $C$ in $Y$. Thus we consider $Y$ to be a partial order wrt cone $C$.

Let for each $i \in I$, an arbitrary set $Y_i$ and $f_i : K \rightarrow Y_i$. Also we define for each $i \in I$, a map $\Psi_i : Y_i \times K_i \times K_i \rightarrow Y$ and $A_i : K \rightarrow 2^{K_i}$ be a multivalued map with nonempty convex values. We put $f = (f_i)_{i \in I}$, $\Psi = (\Psi_i)_{i \in I}$ and define a multivalued map $A(x) = \prod_{i \in I} A_i(x)$.

Now we consider the following vector equilibrium problems over the product set $K$:

\begin{equation}
(\Psi - \text{VEP}) \text{ find } \bar{x} \in K \text{ such that } \bar{x} \in A(\bar{x}) \text{ and } \sum_{i \in I} \Psi_i(f_i(\bar{x}), \bar{x}; y_i) \notin \text{ int} C, \forall y_i \in A_i(\bar{x}), i \in I; \nonumber
\end{equation}

and the Minty type vector equilibrium problem

\begin{equation}
(\Psi - \text{MVEP}) \text{ find } \bar{x} \in K \text{ such that } \bar{x} \in A(\bar{x}) \text{ and } \sum_{i \in I} \Psi_i(f_i(y), \bar{x}; y_i) \notin \text{ int} C, \forall y_i \in A_i(\bar{x}), i \in I. \nonumber
\end{equation}
Also, we consider the Stampacchia type system of vector equilibrium problems:

\[
\Psi \left( f_i(x), x_i, y_i \right) \notin \text{int } C \quad \forall y_i \in A_i(x), \quad i \in I.
\]

We denote by \( K_s, K_s^m \) and \( K_{ss} \) the solution sets of (\( \Psi \)-VEP), (\( \Psi \)-MVEP) and (\( \Psi \)-SVEP), respectively.

In the following we introduce some classes of mappings which extend the ones of relatively pseudomonotony, relatively maximal pseudomonotony and hemicontinuity. Further, some basic results relatively to this classes are stated.

**Definition 2.1.** The family \( \{f_i\}_{i=1}^n \) is

(i) relatively pseudomonotone wrt \( \Psi \) if for all \( x, y \in K \) we have

\[
\sum_{i=1}^n \Psi_i(f_i(x), x_i, y_i) \notin \text{int } C \Rightarrow \sum_{i=1}^n \Psi_i(f_i(y), x_i, y_i) \notin \text{int } C;
\]

(ii) relatively maximal pseudomonotone wrt \( \Psi \) if it is relatively pseudomonotone wrt \( \Psi \) and for all \( x, y \in K \) we have

\[
\sum_{i=1}^n \Psi_i(f_i(z), x_i, z_i) \notin \text{int } C \quad \forall z \in (x, y) \Rightarrow \sum_{i=1}^n \Psi_i(f_i(x), x_i, y_i) \notin \text{int } C,
\]

where \((x, y) = \prod_{i=1}^n (x_i, y_i)\).

**Definition 2.2.** The family \( \{f_i\}_{i=1}^n \) is hemicontinuous wrt \( \Psi \) if for all \( x, y \in K \) and \( \lambda \in [0,1] \), the mapping \( \lambda \mapsto \sum_{i=1}^n \Psi_i(f_i(x + \lambda(y - x)), x_i, y_i) \) is continuous.

Now we consider some relation between the sets \( K_s, K_{ss} \) and \( K_s^m \).

**Lemma 2.1.** We suppose the family \( \{f_i\}_{i=1}^n \) is hemicontinuous wrt \( \Psi \) and for each \( i \in I \), \( \Psi_i(f_i(x), x_i; x_i) = 0 \) for any \( x \in K \). Then \( K_s \subseteq K_{ss} \).

**Proof.** Let \( \bar{x} \in K_s \). Now we see that \( x_i \in A_i(\bar{x}) \) for all \( i \in I \), then \( x = (x_i)_{i=1}^n \in A(\bar{x}) \). Since \( \bar{x} \in A(\bar{x}) \) we have that also \( y \) defined by \( y_i = x_i \) with arbitrarily fixed \( i \in I \) and \( y_j = \bar{x}_j \) for each \( j \neq i \) is an element of \( K_s \). Using hemicontinuity and sequentially substituting \( y \) in (\( \Psi \)-VEP), with \( i = 1, 2, \ldots, n \), we get that \( \bar{x} \) is a solution of (\( \Psi \)-SVEP), i.e, \( \bar{x} \in K_{ss} \) and lemma is proved.

**Lemma 2.2.** We suppose

(i) the family \( \{f_i\}_{i=1}^n \) is relatively maximal pseudomonotone wrt \( \Psi \);

(ii) for each \( i \in I \), \( A_i \) is nonempty and convex-valued map.

Then \( K_s = K_s^m \).

**Proof.** Using the assumption of relatively pseudomonotonicity wrt \( \Psi \), we get easily that \( K_s \subseteq K_{s}^{m} \).

Now let \( \bar{x} \in K_{s}^{m} \). Then \( \bar{x} \in A(\bar{x}) \) and

\[
\sum_{i=1}^n \Psi_i(f_i(y), \bar{x}_i, y_i) \notin \text{int } C \quad \forall y_i \in A_i(\bar{x}).
\]

But we have that \((x_i, y_i) \in A_i(\bar{x})\) for any \( i \in I \). Hence by (2.1) we obtain

\[
\sum_{i=1}^n \Psi_i(f_i(z), \bar{x}_i, z_i) \notin \text{int } C \quad \forall z_i \in [\bar{x}_i, y_i], \quad i \in I
\]
Now using again relatively pseudomonotonicity, we obtain
\[ \sum_{i \in I} \Psi_i(f_i(x), x_i; y_i) \notin \text{int} C, \; \forall y_i \in A(x), \; i \in I, \]
i.e., \( x \in K_s \) or \( K_s^m \subseteq K_s \). Thus the proof is complete.

Definition 2.3 [8]. A subset \( B \) of a topological space \( E \) is said to be compactly open (respectively, compactly closed) in \( E \) if, for any nonempty compact subset \( D \) of \( E \), \( B \cap D \) is open (respectively, closed) in \( D \).

THEOREM 2.1 [8]. Let \( K \) be a nonempty convex subset of a topological vector space (not necessarily Hausdorff) \( E \) and let \( S, T : K \to 2^K \) be multivalued maps. Assume the following conditions hold:

(a) For all \( x \in K \), \( S(x) \subseteq T(x) \);

(b) For all \( x \in K \), \( T(x) \) is convex and \( S(x) \) is nonempty;

(c) For all \( y \in K \), \( S^{-1}(y) = \{ x \in K | y \in S(x) \} \) is compactly open (i.e. for any nonempty compact subset \( D \) of \( E \), \( S^{-1}(y) \cap D \) is open in \( D \));

(d) There exist a nonempty closed compact (not necessarily convex) subset \( D \) of \( K \) and a \( \tilde{y} \in D \) such that \( K \setminus D \subseteq T^{-1}(\tilde{y}) \).

Then, there exists \( \hat{x} \in K \) such that \( \hat{x} \in T(\hat{x}) \).

3. MAIN RESULTS

Let for each \( i \in I \), \( X_i \) be a real topological vector space, \( Y, C, K_j, A_i \) for \( i \in I \), \( K \) and \( A \) defined as in Section 2. Further we assume that for each \( i \in I \) and for all \( y_i \in K_j \), \( A_i^{-1}(y_i) \) is compactly open in \( K \), and the set \( \mathcal{F} = \{ x \in K | x \in A(x) \} \) is compactly closed.

THEOREM 3.1. We assume

(i) the family \( \{ f_i \}_{i \in I} \) is relatively maximal pseudomonotone wrt \( \Psi \);

(ii) there exist a nonempty closed and compact set \( D \) of \( K \) and \( \tilde{y} \in D \) such that
\[ \sum_{i \in I} \Psi_i(f_i(x), x_i; \tilde{y}_i) \notin \text{int} C, \; \forall x \in K \setminus D \text{ with } \tilde{y} \in A(x); \]

(iii) the mapping \( y \mapsto \sum_{i \in I} \Psi_i(f_i(x), x_i; y_i) \) is quasi convex on \( K \) for any \( x \in K \);

(iv) \( \sum_{i \in I} \Psi_i(f_i(x), x_i; x_i) = 0, \; \forall x \in K \).

Then \( K_s \neq \emptyset \) and \( K_{ss} \neq \emptyset \).

Proof. The proof of this theorem is based on Theorem 2.1. In order to do this, we construct two applications \( S \) and \( T \) that satisfy the hypotheses of the above mentioned theorem.

Let the multivalued maps \( S, T : K \to 2^K \) given by
\[ S(x) = \begin{cases} A(x) \cap \left\{ y \in K | \sum_{i \in I} \Psi_i(f_i(y), x_i; y_i) \notin \text{int} C \right\} & \text{if } x \in \mathcal{F} \\ A(x) & \text{if } x \in K \setminus \mathcal{F} \end{cases} \]
and

$$T(x) = \begin{cases} A(x) \cap \left\{ y \in K : \sum_{i \in I} \Psi_i(f_i(x), x_i, y_i) \in -\text{int } C \right\} & \text{if } x \in F, \\ A(x) & \text{if } x \in K \setminus F. \end{cases}$$

If $P, Q : K \to 2^K$ are given by

$$P(x) = \left\{ y \in K : \sum_{i \in I} \Psi_i(f_i(y), x_i, y_i) \in -\text{int } C \right\},$$

and

$$Q(x) = \left\{ y \in K : \sum_{i \in I} \Psi_i(f_i(x), x_i, y_i) \in -\text{int } C \right\},$$

then we have

$$S(x) = \begin{cases} A(x) \cap P(x) & \text{if } x \in F, \\ A(x) & \text{if } x \in K \setminus F \end{cases} \quad \text{and} \quad T(x) = \begin{cases} A(x) \cap Q(x) & \text{if } x \in F, \\ A(x) & \text{if } x \in K \setminus F \end{cases}.$$ 

By $(i_3)$ we get that for each $x \in K$, $Q(x)$ is convex and then by $(i_1)$ we obtain $P(x) \subseteq Q(x)$ for any $x \in K$.

Since for each $y \in K$ the complement of $P^{-1}(y)$ in $K$ is given by

$$\left[ P^{-1}(y) \right]^c = \left\{ x \in K : \sum_{i \in I} \Psi_i(f_i(y), x_i, y_i) \notin -\text{int } C \right\},$$

is a closed set in $K$, we have that the set $P^{-1}(y)$ is an open set in $K$. Thus, the set $P^{-1}(y)$, for any $y \in K$, is a compactly open set.

Also we see that $A(x)$ is a nonempty convex set. Since for any $i \in I$ and $y_i \in K_i$, $A_i^{-1}(y_i)$ is compactly open set, then $A^{-1}(y) = \bigcap_{i \in I} A_i^{-1}(y_i)$ is a compactly open set in $K$ for all $y \in K$.

Thus, for all $x \in K$, $T(x)$ is a convex set with $S(x) \subseteq T(x)$, i.e., $(a_1)$ and the first condition of $(b_1)$ from THEOREM 2.1. According to [9], Lemma 2.3, we have

$$S^{-1}(y) = \left( A^{-1}(y) \cap P^{-1}(y) \right) \cup \left( (K \setminus F) \cap A^{-1}(y) \right).$$

Now, since for each $y \in K$, $A^{-1}(y)$, $P^{-1}(y)$ and $K \setminus F$ are compactly open sets, then the set $S^{-1}(y)$ is compactly open (see [10]), i.e. $(c_1)$ from Theorem 2.1 hold.

Now we prove that there exists $\bar{x} \in F$, $A(\bar{x}) \cap P(\bar{x}) = \emptyset$. In order to do this, we suppose that $A(x) \cap P(x) \neq \emptyset$ for all $x \in F$. Hence $S(x) \neq \emptyset$ for all $x \in K$, i.e., $(b_1)$ from Theorem 2.1 is true. Finally, we observe that $(i_2)$ is equivalent with $(d_2)$ from the same theorem. Therefore we can apply this result. Thus, there exists $x^0 \in K$ such that $x^0 \in T(x^0)$. Because $\{x \in K | x \in T(x)\} \subseteq \{x \in K | x \in A(x)\} = F$, we get $x^0 \in F$ and $x^0 \in A(x^0) \cap Q(x^0)$. But $x^0 \in Q(x^0)$ implies that
\[ \sum_{i \in I} \Psi_i(f_i(x^0), x^0; x^0) \in -\text{int } C, \]

which contradicts \((i_i)\).

Therefore, there exists \( \bar{x} \in \mathcal{F} \) with \( A(\bar{x}) \cap P(\bar{x}) = \emptyset \). This statement is equivalent with \( \bar{x} \in A(\bar{x}) \) and

\[ \sum_{i \in I} \Psi_i(f_i(y), x_i; y_i) \not\in \text{int } C, \quad \forall y_i \in A(\bar{x}), \quad i \in I, \quad \text{i.e.,} \quad \bar{x} \in K^m \quad \text{and by Lemma 2.2 we get} \quad \bar{x} \in K_x. \]

Finally, from the above result and from Lemma 2.1, we obtain \( \bar{x} \in K_{ss} \neq \emptyset \), and the theorem is proved.

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