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CONSERVATION LAWS OF COUPLED KLEIN-GORDON EQUATIONS WITH CUBIC AND POWER LAW NONLINEARITIES

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This paper determines the conservation laws of the coupled Klein-Gordon equations that arise in the quantum field theory. There are two types of nonlinearity that will be considered, namely the cubic law and the power law. The Lie symmetry analysis with multiplier approach will be the mathematical tool that will be adopted to extract the conserved densities. The conservation laws will be finally determined from these conserved densities from the corresponding 1-soliton solution.

Key words: conservation laws, solitons, Lie symmetry, integrability.

1. INTRODUCTION

The theory of nonlinear evolution equations (NLEEs) is a very important topic of research in theoretical physics, applied mathematics and engineering sciences. These NLEEs lay the basic foundation for advanced studies in this field. In fact NLEEs appear in fluid dynamics, plasma physics, nonlinear optics, nuclear physics, and several other research areas [1-23]. While the integrability aspects, perturbation analysis, numerical simulations, stability analysis are the predominant focus areas in a lot of works, this paper is going to address another important feature that is less studied globally. This is the conservation law that is also known as the *integral of motion*. These integrals of motion are, on many occasions, computed with hand calculations and therefore it does not lead to an exhaustive count for NLEEs. This paper is going to use an adequate tool that lists these integrals of motion exhaustively.

The nonlinear Klein-Gordon equation (KGE) is the NLEE that is going to be considered in this paper. KGE is a very important equation in the area of theoretical and mathematical physics. In particular, it is studied in the context of relativistic quantum mechanics. KGE has been studied by several authors and has been exhaustively considered. This paper is going to take a look at the coupled KGE where cubic and power law nonlinearities are being taken into account [14, 15]. The extraction of conservation laws is going to be the focus here. The *Lie symmetry analysis* with the multiplier approach will be the machinery for extraction of the conservation laws. Initially, this approach will yield the conserved densities. Subsequently, the 1-soliton solution will be used to finally obtain the conserved quantities. These 1-soliton solutions were obtained earlier in the literature [14].

2. GOVERNING EQUATIONS

The coupled KGE that are going to be studied in this paper are first listed in the following two subsections. The cubic law as well as the power law nonlinearity are going to be the studied. The following

two subsections will list the dimensionless form of these equations along with their respective 1-soliton solutions. The constraint conditions or the domain restrictions for the existence of the solitons will also be given.

2.1. Cubic nonlinearity

The coupled KGE with cubic law nonlinearity, in dimensionless form is given by [14, 15]:

$$q_{tt} - k^2 q_{xx} + a_1 q + b_1 q^3 + c_1 q r^2 = 0\,, \tag{1}$$

$$r_{tt} - k^2 r_{xx} + a_2 r + b_2 r^3 + c_2 q^2 r = 0\,. \tag{2}$$

Here in (1) and (2), $q(x,t)$ and $r(x,t)$ are the dependent variables and represent the wave profiles. The independent variables are x and t that respectively, represent the spatial and temporal variable. The non-zero real-valued constants are k, a_j, b_j and c_j for $j = 1,2$. The coupling terms are given by the coefficients of c_j. In particular, equations (1) and (2) together form the coupled $\Phi - 4$ equation. The exact 1-soliton solutions to (1) and (2), are [14]:

$$q(x,t) = A_1 \operatorname{sech}[B(x - vt)] \tag{3}$$

and

$$r(x,t) = A_2 \operatorname{sech}[B(x - vt)]\,, \tag{4}$$

where the amplitudes A_j for $j = 1,2$ are

$$A_j = \sqrt{\frac{2a_j\,(c_j - b_{\bar{j}})}{b_j b_{\bar{j}} - c_j c_{\bar{j}}}} \tag{5}$$

and $j = 1,2$ and $\bar{j} = 3 - j$. The width of the soliton is given by:

$$B = \sqrt{\frac{-a_1}{v^2 - k^2}} = \sqrt{\frac{-a_2}{v^2 - k^2}}\,. \tag{6}$$

Additionally, the parameter v represents the soliton velocity in (3) and (4). These relations consequently introduce the constraint conditions

$$a_1 = a_2\,, \tag{7}$$

$$a_j\,(v^2 - k^2) < 0\,, \tag{8}$$

for $j = 1,2$. Also, the amplitude relations dictate the constraint condition given by

$$a_j\,(c_j - b_{\bar{j}})(b_j b_{\bar{j}} - c_j c_{\bar{j}}) > 0\,. \tag{9}$$

2.2. Power law nonlinearity

For power law nonlinearity, the dimensionless form of the coupled KGE is given by [14, 15]

$$q_{tt} - k^2 q_{xx} + a_1 q + b_1 q^{m+n} + c_1 q^m r^n = 0\,, \tag{10}$$

$$r_{tt} - k^2 r_{xx} + a_2 r + b_2 r^{m+n} + c_2 q^n r^m = 0\,, \tag{11}$$

where $n \neq 1$. While the coefficients have the same interpretation as in the previous subsection, the additional parameters in this case are the power law nonlinearity parameters *m* and *n* and here $m > 0$ as well as $n > 0$. In this case, the 1-soliton solutions to (10) and (11) are [9]:

$$q(x,\ t) = A_1 \ \ \mathrm{sech}^{\frac{2}{m+n-1}} \ \ [B(x - vt)] \tag{12}$$

and

$$r(x,\ t) = A_2 \ \ \mathrm{sech}^{\frac{2}{m+n-1}} \ \ [B(x - vt)] \tag{13}$$

where the amplitudes A_j are connected to each other by the coupled relations:

$$2b_j A_j^{m+n-1} + 2c_j A_j^{m-1} A_{\bar{j}}^n + (m+n-1)\, a_j = 0, \tag{14}$$

where $j = 1, 2$ and $\bar{j} = 3 - j$. The width of the solitons is given by:

$$B = \frac{m+n-1}{2}\sqrt{\frac{-a_1}{v^2 - k^2}} = \frac{m+n-1}{2}\sqrt{\frac{-a_2}{v^2 - k^2}}, \tag{15}$$

where *v* is the velocity of the soliton. Equation (15) is valid with the condition $a_j\,(v^2 - k^2) < 0$. Both the amplitude and width relations lead to the constraint conditions given by (7) and (8).

3. SYMMETRIES AND CONSERVATION LAWS

The Lie symmetry approach on differential equations is well known; for details see e.g., [4, 12]. In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence (see [7]). Firstly, if (T^t, T^x) is a conserved vector corresponding to a conservation law, then

$$D_t T^t + D_x T^x = 0 \tag{16}$$

along the solutions of the differential equation (say, $de = 0$).

Moreover, if there exists a nontrivial differential function Q, called a ‘multiplier', such that

$$E_u[Q(de)] = 0,$$

then $Q(de)$ is a total divergence, i.e.,

$$Q(de) = D_t T^t + D_x T^x,$$

for some (conserved) vector (T^t, T^x) and E_u is the respective Euler-Lagrange operator. Thus, a knowledge of each multiplier *Q* leads to a conserved vector determined by, inter alia, a *homotopy operator*; see more details and references in [7, 8].

For a system $de1 = 0$ and $de2 = 0, Q = (Q^1, Q^2)$ say, so that

$$Q^1(de1) + Q^2(de2) = D_t T^t + D_x T^x,$$

and

$$E_{(u,v)}[D_t T^t + D_x T^x] = 0 \tag{17}$$

In each case, T^t is the *conserved density*.

3.1. Cubic nonlinearity

For coupled KGE with cubic nonlinearity, these equations are:

$$\begin{aligned} &q_{tt} - k^2 q_{xx} + a_1 q + b_1 q^3 + c_1 q r^2 = 0, \\ &r_{tt} - k^2 r_{xx} + a_2 r + b_2 r^3 + c_2 r q^2 = 0. \end{aligned} \tag{18}$$

A one parameter Lie group of Lie point transformations (as vector fields) that leave invariant (18) are:

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = t\partial_x + \frac{1}{k^2} x\partial_t,$$

The conserved vectors (fluxes and densities) are listed below:

$$\text{(i)} \quad (Q_1^1,\ Q_1^2) = (k^2 t q_x + x q_t,\ k^2 t r_x + x r_t),$$

$$\begin{aligned} T_1^x = {} & \frac{1}{4c_1} k^2 [b_1 c_2 t q^4 + 2a_2 c_1 t r^2 + b_2 c_1 t r^4 + 2c_2 t q^2 (a_1 + c_1 r^2) - 2(c_2 x q_t q_x + c_2 k^2 t q_x^2 + \\ & + c_1 r_x (x r_t + k^2 t r_x) + 2c_2 q (q_t + t q_{tt} + x q_{xt}) + 2c_1 r (r_t + t r_{tt} + x r_{xt})], \\ T_1^t = {} & \frac{1}{4c_1} [b_1 c_2 x q^4 + 2a_2 c_1 x r^2 + b_2 c_1 x r^4 + 2c_2 x q^2 (a_1 + c_1 r^2) + 2(c_2 x q_t^2 + c_2 k^2 t q_t q_x + \\ & + c_1 r_t (x r_t + k^2 t r_x) - 2c_2 k^2 q (q_x + t q_{xt} + x q_{xx}) - 2c_1 k^2 r (r_x + t r_{xt} + x r_{xx})]; \end{aligned}$$

$$\text{(ii)} \quad (Q_2^1,\ Q_2^2) = (\frac{c_2}{c_1} q_t,\ r_t);$$

$$\begin{aligned} T_2^x = {} & \frac{1}{2c_1} [k^2 (-c_2 q_t q_x - c_1 r_t r_x + c_2 q q_{xt} + c_1 r r_{xt})], \\ T_2^t = {} & \frac{1}{4c_1} [b_1 c_2 q^4 + 2a_2 c_1 r^2 + b_2 c_1 r^4 + 2c_2 q^2 (a_1 + c_1 v^2) + 2c_2 q_t^2 +, \\ & + 2c_1 r_t^2 - 2c_2 k^2 q q_{xx} - 2c_1 k^2 r r_{xx}]; \end{aligned}$$

$$\text{(iii)} \quad (Q_3^1,\ Q_3^2) = (\frac{c_2}{c_1} q_x,\ r_x);$$

$$\begin{aligned} T_3^x = {} & \frac{1}{4c_1} [b_1 c_2 q^4 + 2a_1 c_1 r^2 + b_2 c_1 r^4 + 2c_2 q^2 (a_1 + c_1 v^2) + 2c_2 q_t^2 + \\ & + 2c_2 q q_{tt} + 2c_1 r r_{tt} - 2k^2 (c_2 q_x^2 + c_1 r_x^2)], \\ T_3^t = {} & \frac{1}{2c_1} (c_2 q_t q_x + c_1 r_t r_x - c_2 q q_{xt} - c_1 r r_{xt})]. \end{aligned}$$

3.2. Power law nonlinearity

In the case of power law nonlinearity, the coupled KGE can be written:

$$\begin{aligned} &q_{tt} - k^2 q_{xx} + a_1 q + b_1 q^{m+n} + c_1 q^m r^n = 0, \\ &r_{tt} - k^2 r_{xx} + a_2 r + b_2 r^{m+n} + c_2 r^m q^n = 0. \end{aligned} \tag{19}$$

The conserved densities are, however, obtainable only for $m = n - 1$, and are given by

(i)

$$T_1^t=\int_0^1\left[\frac{a_1c_2\lambda q^2}{c_1}+\frac{b_1c_2\lambda q^{2n}}{c_1}+a_2\lambda r^2+2c_2\lambda^{-1+2n}q^nr^n+b_2\lambda^{-1+2n}r^{2n}+\frac{c_2\lambda q_t^2}{c_1}+\lambda r_t^2-\frac{c_2k^2\lambda qq_{xx}}{c_1}-k^2\lambda rr_{xx}\right]\mathrm{d}\lambda$$

$$=\frac{a_1c_2q^2}{2c_1}+\frac{b_1c_2q^{2n}}{2nc_1}+\frac{1}{2}a_2r^2+\frac{1}{n}c_2q^nr^n+\frac{1}{2n}b_2r^{2n}+\frac{c_2q_t^2}{2c_1}+\frac{1}{2}r_t^2-\frac{c_2k^2qq_{xx}}{2c_1}-\frac{1}{2}k^2rr_{xx},$$

(ii)

$$T_2^t=\frac{1}{2c_1}[c_2q_tq_x+c_1r_tr_x-c_2qq_{xt}-c_1rr_{xt}],$$

(iii)

$$T_3^t=\int_0^1\left[\frac{a_1c_2k^2t\lambda q^2}{c_1}+\frac{b_1c_2k^2t\lambda^{-1+2n}q^{2n}}{c_1}+a_2k^2t\lambda r^2+2c_2k^2t\lambda^{-1+2n}q^nr^n+b_2k^2t\lambda^{-1+2n}r^{2n}+\right.$$

$$+\frac{c_2k^2\lambda qq_t}{c_1}+k^2\lambda rr_t+\frac{c_2k^2\lambda qq_{tt}}{c_1}+k^2t\lambda rr_{tt}-\frac{c_2k^2x\lambda q_tq_x}{c_1}-\frac{c_2k^4t\lambda q^2}{c_1}-k^2x\lambda r_tr_x-k^4t\lambda r_x^2+$$

$$\left.+\frac{c_2k^2x\lambda qq_{xt}}{c_1}+k^2x\lambda rr_{xt}\right]\mathrm{d}\lambda=$$

$$=\frac{a_1c_2k^2tq^2}{2c_1}+\frac{b_1c_2k^2tq^{2n}}{2nc_1}+\frac{1}{2}a_2k^2tr^2+\frac{1}{n}c_2k^2tq^nr^n+\frac{1}{2n}b_2k^2tr^{2n}+\frac{c_2k^2qq_t}{2c_1}+\frac{1}{2}k^2rr_t+$$

$$+\frac{c_2k^2tqq_{tt}}{2c_1}+\frac{1}{2}k^2trr_{tt}-\frac{c_2k^2xq_xq_t}{2c_1}-\frac{c_2k^4tq_x^2}{2c_1}-\frac{1}{2}k^2xr_xr_t-\frac{1}{2}k^4tr_x^2+\frac{c_2k^2xqq_{xt}}{2c_1}+\frac{1}{2}k^2xrr_{xt}.$$

4. CONSERVED QUANTITIES

In this section we will obtain the conserved quantities from the conserved densities that are discussed in the previous section after using the 1- soliton solutions that are discussed in Section 2. The study will be again split into the following two subsections depending on the type of nonlinearity in question.

4.1. Cubic nonlinearity

For cubic law nonlinearity, the respective conserved quantities are

$$\begin{aligned}I_1&=\int_{-\infty}^{\infty}T_1^t\mathrm{d}x=\frac{1}{4c_1}\int_{-\infty}^{\infty}[b_1c_2xq^4+2c_1xr^2+b_2c_1xr^4+2c_2xq^2(a_1+c_1r^2)+\\&+2\{c_2xq_t^2+c_2k^2tq_tq_x+c_1r_t(xr_t+k^2tr_x)\}-2c_2k^2q(q_x+tq_{xt}+xq_{xx})-\\&-2c_1k^2r(r_x+tr_{xt}+xr_{xx})]\mathrm{d}x=0,\end{aligned}\tag{20}$$

$$\begin{aligned}I_2&=\int_{-\infty}^{\infty}T_2^t\mathrm{d}x=\frac{1}{4c_1}\int_{-\infty}^{\infty}\{b_1c_2q^4+2a_2c_1r^2+b_2c_1r^4+2c_2q^2(a_1+c_1r^2)+\\&+2c_2q_t^2+2c_1r_t^2-2c_2k^2qq_{xx}-2c_1k^2rr_{xx}\}\mathrm{d}x=\\&=\frac{1}{3c_1B}\{b_1c_2A_1^4+b_2c_1A_2^4+3a_2c_1A_2^2+3a_1c_2A_1^2+2c_1c_2A_1^2A_2^2+(v^2+k^2)(c_2A_1^2+c_1A_2^2)B^2\}\end{aligned}\tag{21}$$

and

$$\begin{aligned}I_3&=\int_{-\infty}^{\infty}T_3^t\mathrm{d}x=\frac{1}{2c_1}\int_{-\infty}^{\infty}[c_2q_tq_x+c_1r_tr_x-c_2qq_{xt}-c_1rr_{xt}]\,\mathrm{d}x\\&=-\frac{2vB}{3c_1}(c_2A_1^2+c_1A_2^2).\end{aligned}\tag{22}$$

In order to evaluate these conserved quantities, the 1-soliton solutions given by (3) and (4) are used in the conserved densities.

4.2. Power law nonlinearity

In case of KGE with power law nonlinearity, the conserved quantities are given by:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} T_1^t \mathrm{d}x = \frac{1}{2c_1}\int_{-\infty}^{\infty}[a_1c_2q^4 + a_2c_1r^2 + c_2q_t^2 + c_1r_t^2 + k^2(c_2q_x^2 + c_1r_x^2) + \\ &+ \frac{1}{n}(b_1c_2q^{2n} + b_2c_1r^{2n})]\mathrm{d}x = \\ &= \frac{1}{2c_1B}\left\{\left(a_1c_2A_1^2 + a_2c_1A_2^2\right) + \frac{B^2(v^2+k^2)(c_2A_1^2 + c_1A_2^2)}{(n-1)(n+1)} + \frac{2(b_1c_2A_1^{2n} + b_2c_1A_2^{2n})}{n(n+1)}\right\}\times \\ &\times \frac{\Gamma\left(\frac{1}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n-1}+\frac{1}{2}\right)}, \end{aligned} \tag{23}$$

$$I_2 = \int_{-\infty}^{\infty} T_2^t \mathrm{d}x = \frac{1}{2c_1}\int_{-\infty}^{\infty}[c_2q_tq_x + c_1r_tr_x - c_2qq_{xt} - c_1rr_{xt}]\mathrm{d}x = -\frac{vB(c_2A_1^2 + c_1A_2^2)}{c_1(n-1)(n+1)}\frac{\Gamma\left(\frac{1}{n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n-1}+\frac{1}{2}\right)}, \tag{24}$$

$$\begin{aligned} I_3 &= \int_{-\infty}^{\infty} T_3^t \mathrm{d}x = \frac{k^2}{2c_1}\int_{-\infty}^{\infty}\{(a_1c_2 + a_2c_1)(q^2+r^2) + \frac{1}{n}(b_1c_2 + b_2c_1 + 2c_1c_2)t(q^{2n} + q^nr^n + r^{2n}) + \\ &+ (c_2qq_t + c_1rr_t) + t(c_2qq_{tt} + c_1rr_{tt}) - x(c_2q_xq_t + c_1r_xr_t) - \\ &- k^2t(c_2q_x^2 + c_1r_x^2) + x(c_2qq_{xt} + c_1rr_{xt})\}\mathrm{d}x = 0. \end{aligned} \tag{25}$$

Once again, the values of the conserved quantities are obtained from the 1-soliton solution that was discussed in the previous section. Also, it was considered the fact that $m = n-1$.

5. CONCLUSIONS

This paper obtained the conserved quantities for the coupled KGE. Both the cubic law and the power law are considered. There are three different types of conservation laws for each type of nonlinearity studied in this paper. The conserved quantities were obtained from their respective conserved densities from the 1-soliton solutions that were published earlier [14]. The Lie symmetry analysis was employed to achieve this goal. It was observed that for power law nonlinearity the conserved quantities exist only if the exponents are connected by a simple relation. This paper only addressed the 2-coupled KGE, however. In future, this mathematical method will be applied to other forms of the coupled KGE such as the 3-coupled as well as the N-coupled KGE [2]. Those results will be reported in future publications.

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