THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 16, Number 4/2015, pp. 477–483

TRANSLATION HYPERSURFACES AND TZITZEICA TRANSLATION HYPERSURFACES OF THE EUCLIDEAN SPACE

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The determinants of bordered Hessian matrices (also called Allen's matrices) play important roles in many areas of mathematics and economics. In the present paper, we completely classify the translation hypersurfaces (M^n,f) whose Allen's matrices of f are singular in the Euclidean $(n+1)$-space R^{n+1}. Also, we obtain that the translation hypersurfaces satisfying the Tzitzeica condition are only hyperplanes in R^{n+1}. An application of such hypersurfaces to production functions in microeconomics is given.

Key words: Allen's matrix, translation hypersurface, Tzitzeica hypersurface, production function, Cobb-Douglas production function, perfect substitute.

1. INTRODUCTION

Let $f:\mathrm{R}^n \to \mathrm{R},\ f=f(x_1,...,x_n)$ be a twice differentiable function. Then the *Hessian matrix* $\mathbf{H}(f)$ is the symmetric square matrix $\left(f_{x_ix_j}\right)$ of second-order partial derivatives of the function f. The *bordered Hessian matrix* of the function f is given by the $(n+1)\times(n+1)$ square matrix

$$\mathbf{H}^B(f)=\begin{pmatrix} 0 & f_{x_1} & \cdots & f_{x_n} \\ f_{x_1} & f_{x_1x_1} & \cdots & f_{x_1x_n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{x_n} & f_{x_nx_1} & \cdots & f_{x_nx_n} \end{pmatrix}, \tag{1.1}$$

where $f_{x_i}=\dfrac{\partial f}{\partial x_i},\ f_{x_ix_j}=\dfrac{\partial^2 f}{\partial x_i\partial x_j}$ for all $i,j\in\{1,...,n\}$.

The bordered Hessian matrices of functions have important applications in various areas of mathematics and economics. For instance, the bordered Hessian matrix is used to analyze quasi-convexity of the functions. If the signs of the bordered principal diagonal determinants of the bordered Hessian matrix of a function are alternate (resp. negative), then the function is quasi-concave (resp. quasi-convex). For more detailed properties see [4, 11, 14]. By using the bordered Hessian matrices, elasticity of substitution of production functions in economics can be defined. Explicitly, let $f=f(x_1,...,x_n)$ be a production function. Then the *Allen elasticity of substitution* of the ith production variable with respect to the jth production variable is given by

$$A_{ij}(\mathbf{x})=-\frac{(x_1f_1+x_2f_2+...+x_nf_n)\mathbf{H}^B(f)_{ij}}{x_ix_j\left(\det\mathbf{H}^B(f)\right)}\ \left(\mathbf{x}=(x_1,...,x_n)\in\mathrm{R}^n_+,\ i,j\in\{1,...n\},\ i\neq j\right),$$

where $\mathbf{H}^B(f)_{ij}$ is the co-factor of the element $f_{x_i x_j}$ in the determinant of $\mathbf{H}^B(f)$ [17]. The authors call the bordered Hessian matrix $\mathbf{H}^B(f)$ by *Allen's matrix* and $\det \mathbf{H}^B(f)$ by *Allen determinant* in [2].

On the other hand, it is known that a hypersurface M^n of the $(n+1)-$dimensional Euclidean space R^{n+1} is called a *translation hypersurface* if it is the graph of a function of the form:

$$f(x_1,...,x_n) = f_1(x_1) + ... + f_n(x_n),$$

where $f_1,..., f_n$ are functions of class C^∞ [15]. We call $f_1,..., f_n$ the *components* of f, and also denote the translation hypersurface M^n by a pair (M^n, f).

The non-lightlike translation hypersurfaces with constant curvature in (semi)-Euclidean spaces have been classified in [12, 16, 20, 21]. W. Goemans and I. Van de Woestyne [13] proved that every translation lightlike and every homothetical lightlike hypersurface in a semi-Euclidean space is a hyperplane.

A class of surfaces, called *Tzitzeica surfaces* in R^3, has important applications both in mathematics and in physics. In the present paper, we give a classification theorem for translation hypersurfaces satisfying Tzitzeica condition and classify the translation hypersurfaces by using Allen's matrices with applications in microeconomics. Throughout this paper, we assume that $f_1,..., f_n$ are real valued functions of class C^∞ and have non-vanishing first derivatives.

2. BASICS ON HYPERSURFACES IN EUCLIDEAN SPACES

Let M^n be a hypersurface of a Euclidean space R^{n+1}. The *Gauss map* $\nu : M^n \to S^{n+1}$ maps M^n to the unit hypersphere S^n of R^{n+1}. The differential $d\nu$ of the Gauss map ν is the *shape operator* or *Weingarten map*. Denote by T_pM^n the tangent space of M^n at the point $p \in M^n$. Then, for $v, w \in T_pM^n$, the shape operator A_p at the point $p \in M^n$ is defined by

$$g(A_p(v), w) = g(d\nu(v), w),$$

where g is the induced metric tensor on M^n from the Euclidean metric of R^{n+1}.

The determinant of the shape operator A_p is called the *Gauss-Kronocker curvature*. A hypersurface having zero Gauss-Kronecker curvature is said to be *developable*. In this case the hypersurface can be flattened onto a hyperplane without distortion. We remark that cylinders and cones are examples of developable surfaces, but the spheres are not (under any metric).

For a given function $f = f(x_1,..., x_n)$, the *graph* of f is the non-parametric hypersurface of R^{n+1} defined by $\varphi(\mathbf{x}) = (x_1,..., x_n, f(\mathbf{x}))$, for $\mathbf{x} = (x_1,..., x_n) \in \mathrm{R}^n$.

Denote by $\omega = \sqrt{1 + \sum_{i=1}^{n} (f_{x_i})^2}$, where $f_{x_i} = \frac{\partial f}{\partial x_i}$. The unit normal vector field of the graph of f is $\xi = \frac{-1}{\omega}(f_{x_1},..., f_{x_n}, -1)$. The Gauss-Kronecker curvature G of the graph of f is $G = \frac{\det(\mathbf{H}(f))}{\omega^{n+2}}$, where $\mathbf{H}(f)$ is the Hessian matrix of f.

3. PRODUCTION MODELS IN MICROECONOMICS

In microeconomics, a *production function* is a mathematical expression which denotes the relations between the output generated of a firm, an industry or an economy and inputs that have been used. Explicitly, a production function is a map which has non-vanishing first derivatives defined by

$$f\ :\mathrm{R}_+^n \to \mathrm{R}_+,\quad f = f(x_1, x_2, ..., x_n),$$

where f is the quantity of output, n are the number of inputs and $x_1, x_2, ..., x_n$ are the inputs.

For more detailed properties of production functions [5, 8, 22].

A homothetic production function is a production function of the form:

$$f(x_1, ..., x_n) = F(h(x_1, ..., x_n)),$$

where $h(x_1, ..., x_n)$ is homogeneous function of arbitrary given degree and F is a monotonically increasing function. Homothetic functions are production functions whose marginal technical rate of substitution is homogeneous of degree zero (see [6] for details).

In 1928, C. W. Cobb and P. H. Douglas introduced in [8] a famous two-factor production function $Y = bL^k C^{1-k}$, where b presents the total factor productivity, Y the total production, L the labor input and C the capital input. This function is nowadays called *Cobb-Douglas production function*. In its generalized form, the Cobb-Douglas production function may be expressed as $f(\mathbf{x}) = \gamma x_1^{\alpha_1} ... x_n^{\alpha_n}$, where γ is a positive constant and $\alpha_1, ..., \alpha_n$ are nonzero constants. A homothetic production function of form $f(\mathbf{x}) = F(x_1^{\alpha_1} ... x_n^{\alpha_n})$ is called a *homothetic generalized Cobb-Douglas production function* [7].

4. CLASSIFICATION OF TRANSLATION HYPERSURFACES

A translation hypersurface (M^n, f) in R^{n+1} is parametrized by

$$\varphi(\mathbf{x}) = (x_1, ..., x_n, f_1(x_1) + ... + f_n(x_n)), \tag{4.1}$$

for $\mathbf{x} = (x_1, ..., x_n) \in \mathrm{R}^n$.

Next result completely classifies the translation hypersurfaces whose the Allen's matrices of f are singular.

THEOREM 4.1. *Let (M^n, f) be a translation hypersurface of R^{n+1}. Then the Allen's matrix $\mathbf{H}^B(f)$ of f is singular if and only if (M^n, f) is one of the following*:

(a) *An open part of the x_1x_2- plane of R^3, when $n = 2$;*

(b) *A product manifold $\Gamma \times \mathrm{R}^2$, where Γ is a curve lying in x_3x_4- plane of R^4, when $n = 3$;*

(c) *A product manifold $M^{n-2} \times \mathrm{R}^2$, where M^{n-2} is a translation hypersurface in R^{n-1}, $n \geq 4$;*

(d) *A translation hypersurface parametrized by,*

$$\varphi(\mathbf{x}) = \left(x_1, ..., x_n, \sum_{i=1}^{n} (\alpha_i \ln(x_i + \beta_i) + \mu_i) \right),\ n \geq 2,$$

where α_i, $i = \overline{1,n}$, are nonzero constans satisfying $\alpha_1 + ... + \alpha_n = 0$ and β_i, μ_i, $i = \overline{1,n}$, are some constants.

Proof. Let f be a function of class C^∞ given by

$$f(\mathbf{x}) = f_1(x_1) + ... + f_n(x_n), \tag{4.2}$$

for $\mathbf{x} = (x_1, ..., x_n) \in \mathrm{R}^{n+1}$. We first need to calculate the determinant of the Allen's matrix $\mathbf{H}^B(f)$. For that, it follows from (4.2) that

$$f_{x_i}=\frac{\partial f}{\partial x_i}=f_i^{'},\ f_{x_ix_j}=\frac{\partial^2 f}{\partial x_i\partial x_j}=0,\ f_{x_ix_i}=\frac{\partial^2 f}{\partial x_i^2}=f_i^{''},\ i,j\in\{1,...,n\}, i\neq j\ .$$

From (1.1), the determinant of the Allen's matrix of the function given by (4.2) is

$$\det\left(\mathbf{H}^B(f)\right)=\begin{vmatrix} 0 & f_1^{'} & f_2^{'} & ... & f_n^{'} \\ f_1^{'} & f_1^{''} & 0 & ... & 0 \\ f_2^{'} & 0 & f_2^{''} & ... & 0 \\ \vdots & \vdots & \vdots & ... & \vdots \\ f_n^{'} & 0 & 0 & ... & f_n^{''} \end{vmatrix}=-\sum_{j=1}^{n} f_1^{''}...f_{j-1}^{''}\left(f_j^{'}\right)^2 f_{j+1}^{''}...f_n^{''}\ .$$

On the other hand, let $\left(M^n, f\right)$ be a translation hypersurface of R^{n+1} parametrized by

$$\varphi(\mathbf{x})=\left(x_1,...,x_n, f_1(x_1)+...+f_n(x_n)\right),$$

where $f_1(x_1),..., f_n(x_n)$ are functions of class C^∞ having nonzero first derivatives. Let us assume that the Allen's matrix $\mathbf{H}^B(f)$ is singular. Then we have

$$\sum_{j=1}^{n} f_1^{''}...f_{j-1}^{''}\left(f_j^{'}\right)^2 f_{j+1}^{''}...f_n^{''}=0\ . \tag{4.3}$$

From (4.3), two cases occur:

Case (i): At least one of $f_1^{''},..., f_n^{''}$ vanishes. Without loss of generality, we may assume that $f_1^{''}=0$. Hence from (4.3), we get

$$\left(f_1^{'}\right)^2 f_2^{''}...f_n^{''}=0. \tag{4.4}$$

Without loss of generality, we may assume from (4.4) that $f_2^{''}=0.$ Thus we have

$$f_1(x_1)=\alpha_1x_1+\beta_1,\quad f_2(x_2)=\alpha_2x_2+\beta_2.$$

for some constants $\alpha_1,\alpha_2,\beta_1,\beta_2$ with $\alpha_1,\alpha_2\neq 0$. Hence $\left(M^n, f\right)$ is parametrized by

$$\varphi(\mathbf{x})=\left(x_1,...,x_n,\alpha_1x_1+\alpha_2x_2+f_3(x_3)+...+f_n(x_n)\right),$$

which implies cases (a)–(c).

Case (ii): $f_1^{''},..., f_n^{''}$ are nonzero. Then from (4.3), by dividing with the product $f_1^{''}...f_n^{''}$, we write

$$\frac{\left(f_1^{'}\right)^2}{f_1^{''}}+...+\frac{\left(f_n^{'}\right)^2}{f_n^{''}}=0\ . \tag{4.5}$$

Taking the partial derivative of (4.5) with respect to x_i, we obtain

$$2\left(f_i^{''}\right)^2=f_i^{'}f_i^{'''},\ i=1,...,n. \tag{4.6}$$

By solving (4.6), we find $f_i=\alpha_i\ln\left|x_i+\beta_i\right|+\mu_i$, for some nonzero constants α_i satisfying $\alpha_1+...+\alpha_n=0,$ because $\dfrac{\left(f_i^{'}\right)^2}{f_i^{''}}=\alpha_i$ and some constants β_i,μ_i for all $i\in\{1,...,n\}$. Therefore, we complete first part of the proof.

Conversely, it is straightforward to verify that each of the cases (a)-(d) implies that f has vanishing Allen determinant.

Theorem 4.1 is a reformulation (in this special case) of a result obtained in [2] by the same authors. As a consequence of Theorem 4.1, we have the following.

COROLLARY 4.2. *Let* (M^n, f) *be a translation hypersurface of* R_+^{n+1} *such that all components of* f *are non-linear functions. Then the Allen's matrix* $\mathbf{H}^B(f)$ *is singular if and only if, up to suitable translations of* $x_1,...,x_n$, (M^n, f) *is the graph hypersurface of the homothetic function given by* $f(x_1,...,x_n) = \ln(x_1^{\alpha_1}...x_n^{\alpha_n})$, *where* $\alpha_1,...,\alpha_n$ *are nonzero constants with* $\alpha_1 + ... + \alpha_n = 0$.

5. TZITZEICA TRANSLATION HYPERSURFACES

Gheorghe Tzitzeica (Romanian mathematician, 1873–1939) introduced a class of curves, nowadays called *Tzitzeica curves* and a class of surfaces of the Euclidean 3-space, called *Tzitzeica surface*.

A *Tzitzeica surface* is a spatial surface for which the ratio of its Gaussian curvature G and the distance d from the origin to the tangent plane at any arbitrary point of the surface satisfy $G/d^4 = k$ for a constant k. This class of surface is of great interest, having important applications both in mathematics and in physics [22]. The relation between Tzitzeica curves and surfaces is the following: for a Tzitzeica surface with negative Gaussian curvature, the asymptotic lines are Tzitzeica curves [9, 10]. In [22] was given a necessary and sufficient condition for Cobb-Douglas production hypersurface to be a Tzitzeica hypersurface. A *Tzitzeica hypersurface* is a hypersurface satisfying

$$G(\mathbf{x}) = k.d(\mathbf{x})^{n+2}, \tag{5.1}$$

where $G(\mathbf{x})$ denotes the Gauss-Kronocker curvature at a point $\mathbf{x}$, $\mathrm{d}(\mathbf{x})$ is the distance between the origin and the hyperplane tangent to the hypersurface at $\mathbf{x}$ and k is a real constant. It is known that the simplest Tzitzeica hypersurface is described by the equation $x_1 \ldots x_n = \frac{1}{x_{n+1}}$.

The following theorem classifies translation hypersurfaces satisfying Tzitzeica condition.

THEOREM 5.1. *Let* (M^n, f) *be a translation hypersurface in* R^{n+1}. Then (M^n, f) *is a Tzitzeica hypersurface if and only if it is an open part of a hyperplane.*

Proof. Let $D \subset \mathrm{R}^n$ be an open domain, $n \geq 2$. Then a translation hypersurface (M^n, f) in R^{n+1} is parametrized by

$$\varphi(\mathbf{x}) = (x_1,...,x_n, f_1(x_1) + ... + f_n(x_n)), \tag{5.2}$$

for $\mathbf{x} = (x_1,...,x_n) \in D$. One can easily see that the unit normal vector ξ and the Gauss-Kronocker curvature G are given by

$$\xi = \frac{-1}{\omega}(f_1',...,f_n',-1) \tag{5.3}$$

and

$$G = \frac{1}{\omega^{n+2}} \prod_{j=1}^{n} f_j'', \tag{5.4}$$

where $f_j' = \frac{df_j}{dx_j}$, $f_j'' = \frac{d^2 f_j}{dx_j^2}$, for all $j \in \{1,...,n\}$ and $\omega = \sqrt{1 + \sum_{j=1}^{n} (f_j')^2}$.

On the other hand the distance between the origin $O = (0,...,0)$ and the tangent hyperplane to the hypersurface at $\mathbf{x} = (x_1,...,x_n)$ is

$$d=\frac{1}{\omega}\left|\sum_{j=1}^{n}x_j f_j^{'}-f_j\right|. \tag{5.5}$$

We distinguish two cases:

Case (i): $G\neq 0$. Assume that $\left(M^n, f\right)$ satisfies Tzitzeica condition. Then by (5.1), (5.4) and (5.5),

$$\frac{\prod_{j=1}^{n} f_j^{''}}{\left(\sum_{j=1}^{n}x_j f_j^{'}-f_j\right)^{n+2}}=k\,, \tag{5.6}$$

for a real constant k. Taking partial derivative of (5.6) with respect to x_i, we derive

$$f_1^{''}\cdots\overset{i}{\widehat{\ }}\cdots f_n^{''}\left(f_i^{'''}\left(\sum_{j=1}^{n}x_j f_j^{'}-f_j\right)-(n+2)x_i\left(f_i^{''}\right)^2\right)=0,$$

where $\overset{i}{\widehat{\ }}$ denotes absence of ith index. For the last equality, since $G\neq 0$, we get

$$f_i^{'''}\left(\sum_{j=1}^{n}x_j f_j^{'}-f_j\right)=(n+2)x_i\left(f_i^{''}\right)^2. \tag{5.7}$$

In the left-hand side of the equation (5.7), $\sum_{j=1}^{n}x_j f_j^{'}-f_j$ is a non-constant function of variables of $x_1,\ldots,\overset{i}{\widehat{\ }},\ldots,x_n$ since $f_j^{''}\neq 0$ for all $j\in\{1,\ldots,n\}-\{i\}$. However the right-hand side is a function of variable x_i. Thus we obtain $f_i^{'''}=0$ and $f_i^{''}=0$ which means that f_i is linear. This is a contradiction, because $G\neq 0$.

Case (ii): $G=0$. Let us assume that $\left(M^n, f\right)$ is a Tzitzeica hypersurface. By (5.1), we have

$$d(\mathbf{x})=\left(x_1\frac{\mathrm{d}f_1}{\mathrm{d}x_1}-f_1\right)+\ldots+\left(x_n\frac{\mathrm{d}f_n}{\mathrm{d}x_n}-f_n\right)=0\,, \tag{5.8}$$

where $\mathbf{x}=(x_1,\ldots,x_n)\in D$. Taking the partial derivative of (5.8) with respect of x_j, $1\leq j\leq n$, yields

$$x_j\frac{\mathrm{d}^2 f_j}{\mathrm{d}x_j^2}=0\,. \tag{5.9}$$

After solving (5.10), we obtain

$$f_j(x_j)=\alpha_j x_j+\beta_j\,, \tag{5.10}$$

where α_j are nonzero constants and β_j some constants for all $j\in\{1,\ldots,n\}$ such that $\beta_1+\ldots+\beta_n=0$. Combining (5.2) and (5.10) gives that $\left(M^n, f\right)$ is an open part of a hyperplane in R^{n+1}. The converse is easy to verify. It immediately follows from Theorem 5.1.

COROLLARY 5.1. *There is no translation hypersurface with* $G\neq 0$ *satisfying Tzitzeica condition in* R^{n+1}.

On the other hand, in economics, goods that are completely substitutable with each other are called *perfect substitutes*. They may be characterized as goods having a constant marginal rate of substitution. Mathematically, a production function is called a *perfect substitute* if it is of the form

$$f(\mathbf{x})=\sum_{i=1}^{n}\alpha_i x_i$$

for some nonzero constants $\alpha_1,...,\alpha_n$.

Thus Theorem 5.1 yields the following application of our study in microeconomics:

THEOREM 5.2. *Let* (M^n, f) *be a translation hypersurface in* R_+^{n+1}. Then (M^n, f) *is a Tzitzeica hypersurface if and only if it is the graph hypersurface of a perfect substitute.*

ACKNOWLEDGEMENTS

The second author was granted with the "Gheorghe Tzitzeica" Prize of the Romanian Academy in 2014 for the group of papers *Curvature Properties of Complex and Real Metric Manifolds. Applications*, published in 2012. She would like to thank the Romanian Academy for this honor and in particular to Acad. Prof. Dr. Radu Miron, her Ph. D. supervisor, for his continuous encouragement. Her work at this paper was supported by the DAAD Fellowship for University Professors and Researchers A/14/02337 at Technical University Berlin, December 2014-January 2015. She thanks Professor Udo Simon for being her host Professor and his kind hospitality.

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Received January 19, 2015