THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 16, Number 4/2015, pp. 505–512

SYMPLECTIC POTENTIAL, COMPLEX COORDINATES AND HIDDEN SYMMETRIES ON TORIC SASAKI-EINSTEIN SPACES

Vladimir SLESAR [1], Mihai VISINESCU [2], Gabriel Eduard VÎLCU [3,4]

[1] University of Craiova, Department of Mathematics, 13 Al.I. Cuza St., Craiova 200585, Romania
E-mail: slesar.vladimir@ucv.ro
[2] National Institute for Physics and Nuclear Engineering, Department of Theoretical Physics,
Magurele, P.O. Box MG-6, Romania
[3] Petroleum-Gas University of Ploieşti, Department of Cybernetics and Economic Informatics,
39 Bucuresti Ave., Ploiesti 100680, Romania
E-mail: gvilcu@upg-ploiesti.ro
[4] University of Bucharest, Faculty of Mathematics and Computer Science, Research Center in Geometry, Topology and Algebra,
14 Academiei St., Sector 1, Bucharest 060042, Romania
E-mail: gvilcu@gta.math.unibuc.ro
E-mail: mvisin@theory.nipne.ro

In this paper we present a method for extracting the special Killing forms on toric Sasaki-Einstein manifolds using the symplectic potential and Delzant construction. Unlike other methods relaying on direct calculation of complex coordinates, this method remains valid in the general case. In the final part of the paper, this general procedure for toric Sasaki-Einstein manifold is exemplified in the case of the 5-dimensional spaces $T^{1,1}$ and $Y^{p,q}$.

Key words: Killing forms, Sasaki-Einstein spaces, Calabi-Yau manifolds, symplectic potential.

1. INTRODUCTION

The interest in Sasaki-Einstein geometry has arisen due to its importance in AdS/CFT correspondence [7] which relates quantum gravity in certain background to ordinary quantum field theory without gravity. The AdS/CFT correspondence relates the properties of Sasaki-Einstein spaces to properties of superconformal field theories. The most general explicitly known higher-dimensional metrics describing rotating black holes with NUT parameters in an asymptotically AdS spacetime were obtained in [2]. In the so called BPS limit these geometries are related to Sasaki-Einstein manifolds [6].

In this paper we briefly present a method to extract special Killing forms on general toric Sasaki-Einstein spaces. For this purpose we investigate the complex structure of the Calabi-Yau cone associated with the Sasaki-Einstein space making use of the interplay between symplectic and complex approaches of the Kähler toric manifolds.

The description of the Calabi-Yau manifold using toric data allows us to write explicitly the complex coordinates and extract the Killing-Yano tensors. The basic tools are represented by the holomorphic volume form, described here using complex coordinates and symplectic potential, and a remarkable characterization of special Killing forms stated in [11] is presented in Section 2. We note that Killing forms and, more generally, Killing-Yano tensors are related to symmetries and integrability of the geodesic motion. In Section 3 of the paper we present two relevant examples: the Sasaki-Einstein spaces $T^{1,1}$ and $Y^{p,q}$. In the last Section we present the conclusions.

2. SPECIAL KILLING FORMS AND TORIC SASAKI-EINSTEIN SPACES

We consider a C^∞ manifold M. A possible way to generalize the Killing vector fields defined on M is represented by Killing-Yano tensors which are antisymmetric tensors and in component notation satisfy the equation [17]

$$\nabla_{(i}\Psi_{j_1)j_2\ldots j_p}=0\,. \tag{1}$$

A particular class of Killing-Yano tensors is represented by the special Killing forms satisfying the additional equation for some constant c [11]

$$\nabla_X(\Psi)=cX^*\wedge\Psi\,, \tag{2}$$

for any vector field X on M.

A symmetric generalization of the Killing vectors is given by the Stäckel-Killing tensors. In fact, a Stäckel-Killing tensor is a symmetric tensor $K_{i_1\ldots i_r}$ of rank $r>1$, satisfying the generalized Killing equation:

$$\nabla_{(j}K_{i_1i_2\ldots i_r)}=0\,.$$

It is well known that for any geodesic γ with tangent vector $\dot\gamma^i$

$$Q_K=K_{i_1\ldots i_r}\dot\gamma^{i_1}\ldots\dot\gamma^{i_r}$$

is constant along γ. This is the analogue of the conserved quantities associated with Killing vectors.

We recall now that there is an important connection between these two generalizations of the Killing vectors: given two Killing-Yano tensors $\psi^{i_1,\ldots,i_k}$ and $\sigma^{i_1,\ldots,i_k}$ there is a Stäckel-Killing tensor of rank 2 defined by

$$K_{ij}^{(\psi,\sigma)}=\psi_{ii_2\ldots i_k}\sigma_j^{i_2\ldots i_k}+\sigma_{ii_2\ldots i_k}\psi_j^{i_2\ldots i_k}\,.$$

We outline that this result offers a method to generate higher order integrals of motion by identifying the complete set of Killing forms.

We pass now to a remarkable class of manifolds where special Killing forms are known to exist, namely the Sasaki-Einstein manifolds. In the following we give a brief presentation. First we introduce the metric cone $C(M)$ of the manifold M. This is in fact the product manifold $M\times\mathbf{R}_{>0}$, with dim $C(M)=2n$, endowed with the warped metric

$$g_{cone}:=dr^2+r^2g\,.$$

A differentiable manifold M is called *Sasaki manifold* if its metric cone $C(M)$ is a Kähler manifold.

If J represents the complex structure on the cone manifold, then we denote

$$\widetilde{K}:=J\left(r\frac{\partial}{\partial r}\right).$$

Considering the restriction of $\widetilde{K}$ to the submanifold determined by the condition r=1, we obtain the Reeb vector field B on the Sasaki manifold M. The dual 1-form of B on M is denoted by η.

An *Einstein manifold* is a Riemannian manifold (M,g) satisfying the Einstein condition

$$Ric_g=\lambda g\,, \tag{3}$$

for a real constant λ, where Ric_g denotes the Ricci tensor of g.

Einstein manifolds with $\lambda=0$ are called *Ricci-flat manifolds*. A *Sasaki-Einstein manifold* is a Riemannian manifold (M,g) that is both Sasaki and Einstein. We note that in the case of Sasaki-Einstein manifolds one always has (3) with the Einstein constant $\lambda=2(n-1)$. It turns out that a Sasaki manifold M is

Einstein if and only if the metric cone $C(M)$ is Kähler Ricci-flat. Consequently, the metric cone of a Sasaki-Einstein manifold will be a Kähler manifold with flat Ricci tensor, *i.e.* a Calabi-Yau manifold [1].

A *toric Sasaki manifold M* is a Sasaki manifold whose Kähler cone $C(M)$ is a toric Kähler manifold [4].

We can easily construct Killing forms of the Sasaki-Einstein manifold M using the 1-form η [14]. In order to find Killing-Yano tensors which are not related to η we must resort to the Calabi-Yau cone of a Sasaki-Einstein manifold.

In the case of the Calabi-Yau cone, there are *two additional* Killing forms of degree n. To write them explicitly we shall use the volume form V of $C(M)$. The complex volume holomorphic form Ω of the metrical cone $C(M)$ can be introduced by the relation

$$\mathsf{V} = \frac{\mathrm{i}^n}{2^n}(-1)^{n(n-1)/2}\Omega\wedge\overline{\Omega}\,.$$

Note that Ω and $\overline{\Omega}$ are parallel on the metric cone. To extract the corresponding additional Killing forms of the Sasaki-Einstein spaces we make use of the fact that for any p-form ψ on the space M we can define an associated p+1-form ψ^C on the cone $C(M)$ by:

$$\psi^C := r^p dr\wedge\psi + \frac{r^{p+1}}{p+1}d\psi\,.$$

We remark that ψ^C is parallel if and only if ψ is a special Killing-Yano tensor [11].

Therefore, to write explicitly the additional Killing-Yano tensors on a Sasaki-Einstein space it is necessary to find the complex volume holomorphic (n,0) form Ω on the Calabi-Yau cone $C(M)$. As we will see below, a convenient expression of the volume holomorphic form is achieved using complex coordinates. First of all we introduce symplectic coordinates.

Considering the toric $\mathbf{T}^n$ action, assume that this is generated by $\partial/\partial\Phi^i$, Φ^i being the angular coordinates along the orbit of the torus action. In order to obtain the action coordinates y^i the key ingredient is represented by the momentum map

$$\mu = \frac{1}{2}r^2\eta\,.$$

We have the correspondence:

$$y^i = \mu\left(\frac{\partial}{\partial\Phi^i}\right) = \frac{1}{2}\left\langle r^2\eta, \frac{\partial}{\partial\Phi^i}\right\rangle. \tag{4}$$

From symplectic coordinates we pass to complex coordinates $z_i = x^i + \mathrm{i}\Phi^i$, with $\mathrm{i} := \sqrt{-1}$. In these coordinates the metric is written as

$$ds^2 = F_{ij}dx^i dx^j + F_{ij}d\Phi^i d\Phi^j\,. \tag{5}$$

where F_{ij} is the Hessian of the Kähler potential F.

The Kähler potential F is related to the symplectic potential G by the Legendre transform

$$F(x) = \left(y^i\frac{\partial G}{\partial y^i} - G\right)(y = \partial F/\partial x)\,.$$

Therefore F and G are Legendre dual to each other

$$F(x) + G(y) = \sum_j \frac{\partial F}{\partial x_j}\frac{\partial G}{\partial y^j} \text{ at } x^i = \frac{\partial G}{\partial y^i} \text{ or } y^i = \frac{\partial F}{\partial x_i}. \tag{6}$$

Obviously, the above construction is possible using the *symplectic potential G*. This object is obtained using the Delzant construction [4].

A *Delzant polytope* is a convex polytope such that there are *n* edges meeting at each vertex, each edge meeting at the vertex is of the form $1+tu_i$, where $u_i \in \mathbf{Z}^n$, and $\{u_i\}$ can be chosen to form a basis in $\mathbf{Z}^n$. This polytope can be described by the inequalities

$$l_A(y) := \langle y, v_A \rangle \geq 0 \text{, for } 1 \leq A \leq d \text{ ,}$$

where $\{v_A\}$ are inward pointing normal vectors to the facets of the polytope and *d* is the number of facets [4].

It is possible to associate to any Delzant polytope $P \in \mathbf{R}^n$ a close connected symplectic manifold *M*, together with a Hamiltonian $\mathbf{T}^n$ action on the manifold. In fact it can be shown that the polytope turns out to be the image of the momentum map, $P = \mu(M)$.

The symplectic potential *G* can be written in terms of the toric data [9]

$$G = G^{can} + G^{\mathsf{B}} + h. \tag{7}$$

In the above description of *G* the first term is the canonical symplectic potential and it is computed using the corresponding Delzant polytope

$$G^{can} = \frac{1}{2}\sum_A l_A(y)\log l_A(y) \,. \tag{8}$$

The second term is related to the Reeb vector field B. Here we define the affine function $l_{\mathsf{B}} := <\mathsf{B},\cdot>$, $l_\infty := <\sum_A v_A, \cdot>$ and G^{B} is written as

$$G^{\mathsf{B}} = \frac{1}{2}\, l_{\mathsf{B}}(y)\log\, l_{\mathsf{B}}(y) - \frac{1}{2} l_\infty(y)\log l_\infty(y) \,.$$

Finally, as in the general case *G* needs to satisfy the Monge-Ampère equation, a homogeneous function *h* of degree *1* in variables *y* is added.

For a complete determination of the symplectic potential (8) it is necessary to find the Reeb vector B and the function *h* from (7). There are two known procedures to extract the components of the Reeb vector from the toric data. The first procedure is based on the maximization of the central charge (*a*-maximization) used in connection with the computation of the Weyl anomaly in 4-dimensional field theory [5]. The second one is known as volume minimization (or *Z*-minimization) [9].

Using the Ricci-flatness condition, the (*n*,0) holomorphic form of the Ricci-flat metric on the Calabi-Yau cone is [9]

$$\Omega = \exp(x^1 + i\Phi^1) dz^1 \wedge \ldots \wedge dz^n \,. \tag{9}$$

3. TWO EXAMPLES

3.1. SPECIAL KILLING FORMS ON $T^{1,1}$

One of the most familiar examples of homogeneous toric Sasaki-Einstein five-manifold is the space $T^{1,1} = S^2 \times S^3$ endowed with the following metric [8]

$$ds^2(T^{1,1}) = \frac{1}{6}\left(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\right)$$
$$+ \frac{1}{9}\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 .$$

Throughout this section we introduce complex coordinates on $C(T^{1,1})$ using the procedure exposed above.

We start out by considering the 1-form η, which in our particular case can be written as

$$\eta = \frac{1}{3}\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right). \tag{10}$$

It is convenient to work with angle coordinates

$$\begin{aligned}
\Phi^1 &:= 2\nu = \psi, \\
\Phi^2 &:= -\frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 + \nu = -\frac{1}{2}\phi_1 + \frac{1}{2}\phi_2 + \frac{1}{2}\psi, \\
\Phi^3 &:= -\frac{1}{2}\phi_1 - \frac{1}{2}\phi_2 + \nu = -\frac{1}{2}\phi_1 - \frac{1}{2}\phi_2 + \frac{1}{2}\psi.
\end{aligned}$$

Then, considering the momentum map, we get also the action coordinates

$$\begin{aligned}
y^1 &= \frac{1}{6}r^2\left(\cos\theta_1 + 1\right), \\
y^2 &= \frac{1}{6}r^2\left(\cos\theta_2 - \cos\theta_1\right), \\
y^3 &= -\frac{1}{6}r^2\left(\cos\theta_1 + \cos\theta_2\right).
\end{aligned}$$

As explained above, from symplectic action-angle coordinates $\left(y^i, \Phi^i\right)$, we pass to the complex coordinates using the symplectic potential G. Then we need to compute G^{can} using (8). We consider the "inward pointing" normal vectors to the cone

$$v_1 = [1, 1, 1],\ v_2 = [1, 0, 1],\ v_3 = [1, 0, 0],\ v_4 = [1, 1, 0].$$

Note that in the particular case of the conifold $T^{1,1}$, the sum $G^{can} + G^{\mathrm{B}}$ is already a solution of the Monge-Ampère equation. Consequently the function h is not needed anymore [10].

Applying the Legendre transform $x^i = \dfrac{\partial G}{\partial y^i}$ we finally obtain the following patch of complex coordinates

$$\begin{aligned}
z^1 &= 3\log r + \log\sin\theta_1 + \log\sin\theta_2 + \mathrm{i}\psi, \\
z^2 &= \frac{3}{2}\log r + \log\sin\frac{\theta_1}{2} + \log\cos\frac{\theta_2}{2} + \frac{\mathrm{i}}{2}\left(\psi + \phi_1 + \phi_2\right), \\
z^3 &= \frac{3}{2}\log r + \log\sin\frac{\theta_1}{2} + \log\sin\frac{\theta_2}{2} + \frac{\mathrm{i}}{2}\left(\psi - \phi_1 - \phi_2\right).
\end{aligned}$$

For the sake of simplicity in the above relations we have ignored some additive constants.

In our particular framework the equation (9) becomes

$$\Omega = \exp(z^1)dz^1 \wedge dz^2 \wedge dz^3 = r^2 dr \wedge \psi + \frac{r^3}{3}d\psi.$$

Decomposing $\psi = \mathrm{Re}\psi + \mathrm{i}\,\mathrm{Im}\psi$, and again ignoring the multiplicative constants, we get the simple formulas below for the special real Killing forms [13]:

$$\begin{aligned}
\mathrm{Re}\psi &= \cos\psi d\theta_1 \wedge d\theta_2 + \sin\theta_2 \sin\psi d\theta_1 \wedge d\phi_2 - \sin\theta_1 \sin\psi d\theta_2 \wedge d\phi_1 - \sin\theta_1 \sin\theta_2 \cos\psi d\phi_1 \wedge d\phi_2, \\
\mathrm{Im}\psi &= \sin\psi d\theta_1 \wedge d\theta_2 - \sin\theta_2 \cos\psi d\theta_1 \wedge d\phi_2 + \sin\theta_1 \cos\psi d\theta_2 \wedge d\phi_1 - \sin\theta_1 \sin\theta_2 \sin\psi d\phi_1 \wedge d\phi_2.
\end{aligned}$$

3.2. SPECIAL KILLING FORMS ON $Y^{p,q}$

The metric tensor of the 5-dimensional manifold $Y^{p,q}$ (which from the topological point of view is again $S^2 \times S^3$) is more complicated that above and can be written as [8]

$$ds^2(Y^{p,q}) = \frac{1-cy}{6}\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}\left(d\psi - \cos\theta d\phi\right)^2 + \\ + w(y)\left[d\alpha + \frac{ac - 2y + cy^2}{6\left(a - y^2\right)}\left(d\psi - \cos\theta d\phi\right)\right]^2, \tag{11}$$

where

$$w(y) = \frac{2(a-y^2)}{1-cy}, \quad q(y) = \frac{a - 3y^2 + 2cy^3}{a - y^2} \tag{12}$$

and $a = a(p,q)$ is a certain constant depending on the relatively prime positive integers p and q. We note that the constant c can be resized by a diffeomorphism, so in what follows we take $c = 1$.

Now, the dual 1-form η is given by [8]:

$$\eta = -2y\ell d\gamma + \frac{1-y}{3}\left(d\psi - \cos\theta d\phi\right), \tag{13}$$

where $\alpha \equiv \ell\gamma$, with $\ell = \ell(p,q)$ again a constant.

In the sequel we employ the angle coordinates

$$\Phi^1 = \psi, \ \Phi^2 = \phi - \psi, \ \Phi^3 = \frac{l}{2}\phi - \frac{l}{2}\psi + \gamma, \tag{14}$$

where $l = p - q$.

As above, using the momentum map we get the action coordinates

$$\begin{aligned} y^1 &= \frac{r^2}{6}(1-y)(1-\cos\theta), \\ y^2 &= -\frac{r^2}{6}(1-y)\cos\theta + \frac{r^2}{2}l\ell y, \\ y^3 &= -\ell r^2 y. \end{aligned} \tag{15}$$

We next construct the symplectic potential G. Let us consider the toric data for $Y^{p,q}$ [8, 10]:

$$v_1 = [1,-1,-p], \ v_2 = [1,0,0], \ v_3 = [1,-1,0], \ v_4 = [1,-2,-p+q]. \tag{16}$$

Unlike the above case of the homogeneous Sasaki-Einstein manifold $T^{1,1}$, the symplectic potential in the case of $Y^{p,q}$ contains the function h. Consequently we can use the expression (8) but we have to introduce the additional vectors v_5 and v_6 as follows [10]:

$$v_5 := \mathsf{B} - v_1 - v_3, \ v_6 = -v_2 - v_4. \tag{17}$$

Using the Legendre transform, the following complex coordinates are now introduced on $C(Y^{p,q})$:

$$
\begin{aligned}
z^1 &= \log\left(r^3\sin\theta\sqrt{y^3-\frac{3}{2}y^2+\frac{a}{2}}\right)+\mathrm{i}\psi,\\
z^2 &= -\log\left(r^3\cos^2\frac{\theta}{2}\sqrt{y^3-\frac{3}{2}y^2+\frac{a}{2}}\right)+\mathrm{i}(\phi-\psi),\\
z^3 &= \log\frac{r^{\frac{p(y_1-y_3)}{1-y_1}}(y-y_3)^{\frac{p(1-y_3)}{2(1-y_1)}}}{\left(\cos\frac{\theta}{2}\right)^l\sqrt{(y-y_1)^p}}+\mathrm{i}\left(\frac{l}{2}\phi-\frac{l}{2}\psi+\gamma\right).
\end{aligned}
\tag{18}
$$

The additive constants are again ignored in the above relations.

After computation, the real special Killing forms $\mathrm{Re}\psi$ and $\mathrm{Im}\psi$ can be written as [14]:

$$
\begin{aligned}
\mathrm{Re}\psi = -\frac{3}{2\ell}\sqrt{\frac{1-y}{6p(y)}}\Big[&\cos\psi\left(d\theta\wedge dy-p(y)\sin\theta d\psi\wedge d\phi+6p(y)\sin\theta d\phi\wedge d\alpha\right)-\\
&-\sin\psi\left(-p(y)d\theta\wedge d\psi-6p(y)d\theta\wedge d\alpha-\sin\theta dy\wedge d\phi+p(y)\cos\theta d\theta\wedge d\phi\right)\Big]
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
\mathrm{Im}\psi = -\frac{3}{2\ell}\sqrt{\frac{1-y}{6p(y)}}\Big[&\sin\psi\left(d\theta\wedge dy-p(y)\sin\theta d\psi\wedge d\phi+6p(y)\sin\theta d\phi\wedge d\alpha\right)+\\
&+\cos\psi\left(-p(y)d\theta\wedge d\psi-6p(y)d\theta\wedge d\alpha-\sin\theta dy\wedge d\phi+p(y)\cos\theta d\theta\wedge d\phi\right)\Big].
\end{aligned}
\tag{20}
$$

Note that the special Killing forms $\mathrm{Re}\psi$ and $\mathrm{Im}\psi$ obtained in this article agree with the results previously obtained in [12, 15] with different approaches.

4. CONCLUSIONS

Special Killing forms and, more generally, Killing-Yano tensors are known to be highly relevant for both mathematics and physics, as they can be used to construct higher order integrals of motion. In this paper we overview the possibility to extract special Killing forms on toric Sasaki-Einstein spaces using successively the symplectic potential, complex coordinates and holomorphic volume form. This approach is possible in the context of the Delzant description of toric geometries. We also remark that for the spaces $T^{1,1}$ and $Y^{p,q}$ the construction of the corresponding Stäckel-Killing tensors from Killing forms allows us to find all conserved quantities for geodesic motions in these spaces. A detailed analysis shows that the number of functionally independent constants of motion is five, which is exactly the number of the degrees of freedom. Therefore these systems are completely integrable facilitating the separation of variables of Hamilton-Jacobi, Klein-Gordon, and Dirac equations.

Finally, we note that the toric geometries could provide new perspectives in investigation of hidden symmetries of other Sasaki-Einstein manifolds of current interest [3, 12, 16].

ACKNOWLEDGEMENTS

The work of VS was supported by the UE grant FP7-PEOPLE-2012-IRSES-316338. MV was supported by CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0137. The work of GEV was supported by CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0118.

REFERENCES

1. C. BOYER, K. GALICKI, *Sasakian geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.
2. W. CHEN, H. LÜ, C.N. POPE, *General Kerr-Nut-AdS metrics in all dimenssions*, Class. Quantum Grav., **23**, p. 5323, 2006.
3. C. VAN COEVERING, *Some examples of toric Sasaki-Einstein manifolds Riemannian topology and geometric structures on manifolds*, Progr. Math., **271**, pp. 185–232, 2009; arXiv:math/0703501.
4. V. GUILLEMIN, *Kaehler structures on toric varieties*, J. Diff. Geom., **40**, pp. 285–309, 1994.
5. K. INTRILIGATOR, B. WECHT, *The exact superconformal R-symmetry maximizes a*, Nucl. Phys. B, **667**, pp. 183–200, 2003.
6. D. KUBIZŇÁK, *On the supersymmetric limit of Kerr-NUT-AdS metrics*, Phys. Lett. B, **675**, p. 110, 2009.
7. J. M. MALDACENA, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys., **2**, pp. 231–252, 1998.
8. D. MARTELLI, J. SPARKS, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, Commun. Math. Phys., **262**, pp. 51–89, 2006; arXiv:hep-th/0411238.
9. D. MARTELLI, J. SPARKS, S.-T. YAU, *The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds*, Commun. Math. Phys., **268**, pp. 39–65, 2006; arXiv:hep-th/0503183.
10. T. OOTA, Y. YASUI, *Toric Sasaki-Einstein manifolds and Heun equations*, Nucl. Phys. B, **742**, pp. 275–294, 2006; arXiv:hep-th/0512124.
11. U. SEMMELMANN, *Conformal Killing forms on Riemannian manifolds*, Math. Z., **245**, pp. 503–527, 2003; arXiv:math.DG/0206117.
12. V. SLESAR, M. VISINESCU, G.E. VÎLCU, *Special Killing forms on Sasaki-Einstein manifolds*, Phys. Scripta, **89**, p. 125205, 2014; arXiv:1403.1015.
13. V. SLESAR, M. VISINESCU, G.E. VÎLCU, *Hidden symmetries on toric Sasaki-Einstein spaces*, EPL, **110**, p. 31001, 2015.
14. V. SLESAR, M. VISINESCU, G.E. VÎLCU, *Toric data, Killing forms and complete integrability of geodesics in Sasaki-Einstein spaces* $Y^{p,q}$, submitted; arXiv:1506.04483.
15. M. VISINESCU, *Killing forms on the five dimensional Einstein-Sasaki* $Y^{p,q}$ *spaces*, Mod. Phys. Lett., **27**, p. 1250217, 2012; arXiv:1207.2581.
16. M. VISINESCU, *Killing forms on 5D Sasaki-Einstein spaces*, Rom. J. Phys., **58**, pp. 675–682, 2013.
17. K. YANO, *Some remarks on tensor fields and curvature*, Ann. Math., **55**, pp. 328–347, 1952.

Received June 18, 2015