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MULTIPLE KINK SOLUTIONS FOR THE SECOND HEAVENLY EQUATION AND THE ASYMMETRIC HEAVENLY EQUATION

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Abstract. In this work, we study the second Plebañski heavenly equation that describes self-dual gravitational fields. We also study an asymmetric heavenly equation that gives an alternative description of anti-self-dual gravity and admits partner symmetries. The two equations are integrable. We determine multiple kink solutions, with distinct physical structures, for each equation. The architectures of the simplified Hirota's method is implemented in this paper. The constraint conditions that fall out which must remain valid in order for the multiple kink solutions to exist are derived.

Key words: second heavenly equation, asymmetric heavenly equation, dispersion relation.

1. INTRODUCTION

In the context of integrable equations, research works are flourishing due to the fact that these equations describe the real features and physical properties in a variety of science, technology, and engineering applications. The basic features of integrable models are the existence of bi-Hamiltonian and the existence of Lax pair for these models. Other different but related criteria methods are used to show the integrability of nonlinear models such as the Painlevé test, the inverse scattering transform method, existence of Darboux transformations, and existence of infinitely many symmetries and conservation laws.

Plebañski introduced the *second four-dimensional (4D) heavenly equation* [1-10]

$$u_{tx}+u_{zy}+u_{xx}u_{yy}-u_{xy}^2=0, \tag{1}$$

which describes *self-dual gravitational fields*. Plebañski introduced heavenly equations for a single potential generating self-dual heavenly metrics which satisfy complex vacuum Einstein equations [4]. In [2], a definition was given to test integrability of such 4D equations. In other words, it was stated that if a $(d+1)$-dimensional partial differential equation (PDE) is said to be integrable if its n-component reductions are locally parameterized by $(d-1)n$ arbitrary functions of one variable. Based on this fact, the second heavenly equation (1) is integrable [2].

A classification of second-order PDEs with four independent variables that possess partner symmetries was given by Sheftel and Malykh in [4]. The asymmetric heavenly equation appears there as one of the canonical equations admitting partner symmetries [3]. The asymmetric 4D heavenly equation reads

$$u_{tx}u_{ty}-u_{tt}u_{xy}+au_{tz}+bu_{xz}+cu_{xx}=0,\ \ a,c\neq 0, \tag{2}$$

where $u(x,y,z,t)$ is the unknown function that depends on the four independent variables t,x,y,z, and $a,b,$ and c are constants. As shown by Dunajski [6], the asymmetric heavenly equation (2) should give an alternative description of anti-self-dual gravity. For $b=0$, Eq. (2) becomes the so-called evolution form of the second heavenly equation.

The second heavenly equation (1) and the asymmetric heavenly equation (2) were studied in Refs. [1–10] and the references therein. The concerns of these works were on developing Hamiltonian structures and the Lax pairs for these two equations and to show that both models are integrable. Moreover, recursion operators, symmetries pseudogroups, deformations, and partner symmetries were examined in the aforementioned references.

In the 1980's, the self-dual equations revolutionized the study of smooth dual equations over different manifolds. These equations are usually called the 4D equations. However, the variable t was used as a time variable to derive multiple kink solutions [1–10]. The second heavenly equation and the asymmetric heavenly equation are self-dual and anti self-dual, respectively. The usual terminology of the solutions of these equations is that all solutions are *solitons*. However, those that are maximal co-dimension, so they are localized in Euclidean space and time are also called *instantons*, or previously called *pseudogroups*. Instantons have many similarities to the lump solitons found in certain sigma models.

Our aim is to study the systems proposed above, namely (1) and (2), motivated by the fact that these models are integrable and are used in Yang-Mills models. We plan to derive multiple-kink solutions for these self-dual and anti-self-dual equations. The *Hirota's direct method*, that is well known now and can be found in Refs. [11–22], will be used to achieve this goal.

2. THE SECOND HEAVENLY EQUATION

In this section we apply the Hirota's direct method to study the second heavenly equation

$$u_{tx}+u_{zy}+u_{xx}u_{yy}-u_{xy}^2=0. \tag{3}$$

Substituting

$$u(x,y,z,t)=e^{\theta_i},\ \ \theta_i=k_ix+r_iy+s_iz-\omega_it, \tag{4}$$

into the linear terms of Eq. (3), and solving the resulting equation for ω_i we obtain the dispersion relation

$$\omega_i=\frac{r_is_i}{k_i},\ \ i=1,2,\cdots N, \tag{5}$$

and hence the wave variable θ_i becomes

$$\theta_i=k_ix+r_iy+s_iz-\frac{r_is_i}{k_i}t. \tag{6}$$

We next use the transformation

$$u(x,y,z,t)=R\big(\ln f(x,y,z,t)\big)_x, \tag{7}$$

where the auxiliary function $f(x,y,z,t)$ is given by

$$f(x,y,z,t)=1+e^{k_1x+r_1y+s_1z-\frac{r_1s_1}{k_1}t}, \tag{8}$$

into Eq. (3) and solve it to find that R can be any real number, hence we select

$$R=1. \tag{9}$$

Substituting Eq. (8) into Eq. (7) gives the single-kink solution

$$u(x,y,z,t)=\frac{k_1e^{k_1x+r_1y+s_1z-\frac{r_1s_1}{k_1}t}}{1+e^{k_1x+r_1y+s_1z-\frac{r_1s_1}{k_1}t}}. \tag{10}$$

Notice that the one-kink solution is obtained for free parameters k_1, r_1, and s_1. This is not the case for the two- and three-kink solutions.

We found that the two- and three-kink solutions cannot be obtained for free parameters as presented for the one-kink solution, instead we have to set

$$r_i=\alpha k_i, \tag{11}$$

where k_i and s_i are left as free parameters, and α is a constant. The last results give the dispersion relation (5) to

$$\omega_i = \alpha s_i,\ i = 1, 2, 3, \cdots N, \tag{12}$$

and hence θ_i becomes

$$\theta_i = k_i x + \alpha k_i y + s_i z - \alpha s_i t. \tag{13}$$

For the two-kink solutions, we substitute

$$u(x, y, z, t) = 2(\ln f(x, y, z, t))_x, \tag{14}$$

where the auxiliary function $f(x, y, z, t)$ for the two-kink solutions is given by

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}, \tag{15}$$

into Eq. (3), where θ_1 and θ_2 for this case are given in (13), to obtain

$$a_{12} = 0, \tag{16}$$

and hence

$$a_{ij} = 0. \tag{17}$$

This shows that the soliton solution of the second heavenly equation is of kink type. The two-kink solutions are thus given by

$$u(x, y, z, t) = \frac{k_1 e^{k_1 x + \alpha k_1 y + s_1 z - \alpha s_1 t} + k_2 e^{k_2 x + \alpha k_2 y + s_2 z - \alpha s_1 t}}{1 + e^{k_1 x + \alpha k_1 y + s_1 z - \alpha s_1 t} + e^{k_2 x + \alpha k_2 y + s_2 z - \alpha s_2 t}}. \tag{18}$$

To determine the three-kink solutions, we substitute the auxiliary function

$$f(x, y, z, t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \tag{19}$$

to obtain

$$u(x, y, z, t) = \frac{\sum_{i=1}^{3} k_i e^{k_i x + \alpha k_i y + s_i z - \alpha s_i t}}{1 + \sum_{i=1}^{3} e^{k_i x + \alpha k_i y + s_i z - \alpha s_i t}}. \tag{20}$$

3. THE ASYMMETRIC HEAVENLY EQUATION

In this section we apply the Hirota's method to study the asymmetric heavenly equation

$$u_{tx} u_{ty} - u_{tt} u_{xy} + a u_{tz} + b u_{xz} + c u_{xx} = 0,\ a, c \neq 0. \tag{21}$$

Substituting

$$u(x, y, z, t) = e^{\theta_i},\ \theta_i = k_i x + r_i y + s_i z - \omega_i t, \tag{22}$$

into the linear terms of Eq. (21), and solving the resulting equation for ω_i we obtain the dispersion relation

$$\omega_i = \frac{b k_i s_i + c k_i^2}{a s_i},\ i = 1, 2, \cdots N, \tag{23}$$

and hence the wave variable θ_i becomes

$$\theta_i = k_i x + r_i y + s_i z - \frac{b k_i s_i + c k_i^2}{a s_i} t. \tag{24}$$

We next use the transformation

$$u(x,y,z,t) = R\left(\ln f(x,y,z,t)\right)_x, \tag{25}$$

where the auxiliary function $f(x,y,z,t)$ is given by

$$f(x,y,z,t) = 1 + e^{k_1 x + r_1 y + \beta k_1 z - \frac{bk_1 s_1 + ck_1^2}{as_1} t}, \tag{26}$$

into Eq. (21) and solve it to find that R can be any real number, hence we select

$$R = 1. \tag{27}$$

Substituting Eq. (26) into Eq. (25) gives the single-kink solution

$$u(x,y,z,t) = \frac{k_1 e^{k_1 x + r_1 y + s_1 z - \frac{bk_1 s_1 + ck_1^2}{as_1} t}}{1 + e^{k_1 x + r_1 y + s_1 z - \frac{bk_1 s_1 + ck_1^2}{as_1} t}}. \tag{28}$$

Notice that the single-kink solution is obtained for free parameters k_1, r_1, and s_1. This is not the case for the two- and three-kink solutions.

We found that the two- and three-kink solutions cannot be obtained for free parameters as constructed for the one-kink solution, instead we have to set

$$s_i = \beta k_i, \tag{29}$$

where k_i and r_i are left as free parameters, and β is a constant. The last results give the dispersion relation (23) to

$$\omega_i = \frac{b\beta + c}{a\beta} k_i, \quad i = 1,2,3,\cdots N, \tag{30}$$

and hence the wave variable θ_i becomes

$$\theta_i = k_i x + r_i y + \beta k_i z - \frac{c + b\beta}{a\beta} k_i t. \tag{31}$$

Following the discussion presented in the previous section, we can show that the soliton solutions possess a kink structure.

For the two-kink solutions, we substitute

$$u(x,y,z,t) = (\ln f(x,y,z,t))_x, \tag{32}$$

where the auxiliary function $f(x,y,z,t)$ for the two-kink solutions is given by

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} \tag{33}$$

into Eq. (21), where θ_1 and θ_2 for this case are given in (31), to obtain the two-kink solutions as

$$u(x,y,z,t) = \frac{k_1 e^{k_1 x + r_1 y + \beta k_1 z - \frac{c+b\beta}{a\beta} k_1 t} + k_2 e^{k_2 x + r_2 y + \beta k_2 z - \frac{c+b\beta}{a\beta} k_2 t}}{1 + e^{k_1 x + r_1 y + \beta k_1 z - \frac{c+b\beta}{a\beta} k_1 t} + e^{k_2 x + r_2 y + \beta k_2 z - \frac{c+b\beta}{a\beta} k_2 t}}. \tag{34}$$

Substituting the auxiliary function

$$f(x,y,z,t) = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \tag{35}$$

gives the three-kink solutions

$$u(x,y,z,t) = \frac{\sum_{i=1}^{3} k_i e^{k_i x + r_i y + \beta k_i z - \frac{c+b\beta}{a\beta} k_i t}}{1 + \sum_{i=1}^{3} e^{k_i x + \alpha k_i y + \beta k_i z - \frac{c+b\beta}{a\beta} k_i t}}. \tag{36}$$

4. DISCUSSION

In this work, we examined the second Plebañski heavenly equation, that describes self-dual gravitational fields, and an asymmetric heavenly equation, that gives an alternative description of anti-self-dual gravity and admits partner symmetries. The aforementioned equations are integrable and have local symmetry pseudo groups. These equations are usually called the 4D equations. However, we used the variable t as a time variable to derive multiple-kink solutions. The usual terminology is that all solutions of these equations are solitons. However, those that are maximal co-dimension, so they are localized in Euclidean space and time are also called instantons. The dispersion relations for the models were found to be distinct. Multiple-kink solutions were formally derived by using the Hirota's method for each model. It is worth noting that for the second heavenly equation, the single-kink solution exists for free parameters, whereas the two- and three-kink solutions exist only for free k_i and s_i, the coefficients of the variables x and z, respectively. However, for the asymmetric heavenly equation, the single-kink solution exists also for free parameters, whereas the two- and three-kink solutions exist only for free k_i and r_i, the coefficients of the variables x and y, respectively. The obtained results are going to be profoundly interesting and useful and will provide a foundation stone for further research activities in this field.

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