THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 17, Number 4/2016, pp. 287–292

UPER AND LOWER BOUNDS FOR A FINITE-TYPE RUIN PROBABILITY IN A NONHOMOGENEOUS RISK PROCESS

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Abstract. Based on many numerical examples, Răducan et al [8] stated a conjecture that relates the order in which some nonhomogeneous claims arrive - provided that all the claims are comparable - to the magnitude of the corresponding ruin probability. In that conjecture the usual stochastic order has been considered for the claims. Now we know that the conjecture was wrong [10] and we prove it for a stronger order, namely the likelihood ratio order. Being stronger, the likelihood order implies the usual stochastic one, but for many usual distributions the two types of ordering are equivalent. That explains our initial conjecture suggested by computer.

Key words: ruin probability, non-homogeneous claims, stochastic orders, risk process, likelihood ratio domination.

1. STATING OF THE PROBLEM

Among various kinds of stochastic ordering on the positive half-line, the strongest one seems to be the likelihood ratio (or, shortly, likelihood). If F_1 and F_2 are two distributions absolutely continuous with respect to some basic measure μ, we say that $F_1 \leq_{like} F_2$ if we can choose the densities f_i for F_i, $i = 1,2$, such that the ratio $\frac{f_1}{f_2}$ is decreasing, with the convention that the ratio makes sense only on the union of the supports of F_1 and F_2, and that $\frac{f_1}{0} = \infty$. We shall use the same notation for random variables: $X_1 \leq_{like} X_2$ means that if $F_1 \leq_{like} F_2$ if F_1 is the distribution of X_1 and F_2 is the distribution of X_2.

In general, this ordering implies the usual stochastic ordering, denoted $F_1 \leq_{st} F_2$ and defined by $F_1(x) \geq F_2(x)$ for all x or, equivalently, by $\overline{F_1}(x) \leq \overline{F_2}(x), \forall x$; here $\overline{F} = 1 - F$ is the right tail of F. However, in some cases, the likelihood ordering is the same with the stochastic one, as we shall see in the following (for more details on stochastic orderings see, e.g., [1, 11]). For this purpose, let Exp(a) denote the exponential distribution with parameter $a > 0$, Gamma (a, b) the gamma distribution with parameters $a,b > 0$, $U(a,b)$ the uniform distribution on (a,b), $a < b$, δ_x the Dirac point measure, and denote by $*$ the convolution operator. Then, for example, in the exponential case, if $F_1 = \text{Exp}\,(a)$ and $F_2 = \text{Exp}\,(b)$, a, $b > 0$, it is easy to see that $F_1 \leq_{like} F_2 \Leftrightarrow F_1 \leq_{st} F_2$ ($\Leftrightarrow a \geq b$). The same equivalence $F_1 \leq_{like} F_2 \Leftrightarrow F_1 \leq_{st} F_2$ holds if $F_1 = \delta_\alpha * \text{Exp}(a), F_2 = \delta_\beta * \text{Exp}(b)$, being also equivalent with ($\alpha \leq \beta$ and $a \geq b$); or, if $F_1 = \text{Gamma}(m,a), F_2 = \text{Gamma}(n,b)$, being equivalent with ($m \leq n$ and $a \geq b$); or, if $F_1 = U(a,b), F_2 = U(a',b')$ which is equivalent with ($a \leq a'$ and $b \leq b'$).

This is why in [8], after many numerical examples (see, also, [7]), we were led by the computer to the following:

CONJECTURE. *Let* $\boldsymbol{X}=(X_k)_{1\le k\le n}, \boldsymbol{Y}=(Y_k)_{1\le k\le n}$ *be independent random vectors valued in* $\mathbb{R}_+^n$ *with independent components, representing the claim sizes and, respectively, the inter-claim revenues of an insurance risk process. Suppose that the components of* $\boldsymbol{Y}$ *are identically distributed and that* $X_1 \le_{st} X_2 \le_{st} \ldots \le_{st} X_n$. *Then* $\Psi_e \le \Psi_\sigma \le \Psi_\varpi$, *where* $\Psi_e, \Psi_\sigma, \Psi_\varpi$ *are the ruin probabilities at or before the* n^{th} *claim when the claims come in the order* $(X_k)_{1\le k\le n}, \left(X_{\sigma(k)}\right)_{1\le k\le n}$ *and respectively,* $(X_{n-k+1})_{1\le k\le n}$, *with σ being an arbitrary permutation of* $\{1,2,\ldots,n\}$.

We discovered [10] that the conjecture is wrong, though it holds for exponentially distributed claims with all parameters distinct. We were misled by the fact that the likelihood ordering is the same with the stochastic one for the distributions we used. In this paper, we prove the correct assertion: the claims should be ordered in the likelihood order. It follows that these claims can follow different types of distributions, which was not the case with the claims considered in [7–9] and [14]; there, the claims were assumed to follow the same type of distribution, but with possible different parameters which implies the non - homogeneity of the risk process. Non-homogeneous claims have already been considered in the actuarial literature in order to capture the fluctuations of the economic environment, see, e.g., De Kok [4], Lefevre and Picard [5], Paulsen [6], Blazevicius *et al*. [2], Castaner *et al*. [3], or Stanford *et al*. [12] and [13].

The structure of the paper is as follows: in Section 2, we present two preliminary results which maybe are interesting in themselves and in Section 3 we prove the conjecture if $X_1 \le_{st} X_2 \le_{st} \ldots \le_{st} X_n$ is replaced by $X_1 \le_{like} X_2 \le_{like} \ldots \le_{like} X_n$.

2. PRELIMINARY RESULTS

The following lemmas are needed to prove the main results:

LEMMA 2.1. *Let* $f:\mathbb{R}_+^2 \to \mathbb{R}$ *be an integrable function satisfying the following properties*:

i) $f(x,y)+f(y,x)=0$ *for any* $x,y \ge 0$,

ii) *if* $x \le y$ *then* $f(x,y) \ge 0$.

Let $\alpha,\beta>0$ *and* $D_{\alpha,\beta}(f)=\int_0^\alpha \int_0^{\alpha+\beta-x} f(x,y)\,\mathrm{d}y\mathrm{d}x$. *Then* $D_{\alpha,\beta}(f)\ge 0$.

Proof. There are two cases, depending if $\alpha>\beta$ or otherwise.

Case 1. $\alpha>\beta$: we rewrite $D_{\alpha,\beta}(f)=\int 1_B f\mathrm{d}\lambda$, , where λ is the Lebesgue measure on $\mathbb{R}$, and 1_B is the indicator function of B. Note that $B=B_1\cup B_2$, where

$$B_1 = \{(x,y)\in\mathbb{R}_+^2 \,|\, x\le\alpha, y\le\alpha, x+y\le\alpha+\beta\},$$

$$B_2 = \{(x,y)\in\mathbb{R}_+^2 \,|\, x\le\alpha, y>\alpha, x+y\le\alpha+\beta\}.$$

Then $D_{\alpha,\beta}(f)=\int 1_{B_1} f\mathrm{d}\lambda+\int 1_{B_2} f\mathrm{d}\lambda.$

The set B_1 is symmetrical, meaning that $(x,y)\in B_1 \Leftrightarrow (y,x)\in B_1$, while the set B_2 can be rewritten as $B_2 = \{(x,y)\in\mathbb{R}_+^2 \,|\, x\le\beta, y>\alpha, x+y\le\alpha+\beta\}$. Therefore, interchanging x and y, applying the assumption (i) and the symmetry of B_1, followed by Fubini's theorem, we obtain

$$\int 1_{B_1} f\,\mathrm{d}\lambda=\int_0^\infty\int_0^\infty 1_{B_1}(x,y)f(x,y)\mathrm{d}x\mathrm{d}y=\int_0^\infty\int_0^\infty 1_{B_1}(y,x)f(y,x)\mathrm{d}y\mathrm{d}x$$

$$=-\int_0^\infty\int_0^\infty 1_{B_1}(y,x)f(x,y)\mathrm{d}x\mathrm{d}y=-\int_0^\infty\int_0^\infty 1_{B_1}(x,y)f(x,y)\mathrm{d}y\mathrm{d}x=-\int 1_{B_1} f\mathrm{d}\lambda,$$

from where $\int 1_{B_1} f \mathrm{d}\lambda = 0$. On the other hand, $\int 1_{B_2} f \mathrm{d}\lambda = \int_0^{\beta} \int_{\alpha}^{\alpha+\beta-x} f(x,y)\mathrm{d}y\mathrm{d}x \geq 0$, since in this case $y \geq \alpha > \beta \geq x$ and according to assumption ii), when $x \leq y$ then $f(x,y) \geq 0$. It follows that $D_{\alpha,\beta}(f) \geq 0$.

Case 2. $\alpha \leq \beta$: now we write $D_{\alpha,\beta}(f) = \int_0^{\alpha}\int_0^{\alpha} f(x,y)\mathrm{d}y\mathrm{d}x + \int_0^{\alpha}\int_0^{\alpha+\beta-x} f(x,y)\mathrm{d}y\mathrm{d}x$. The first integral is again equal to 0 by symmetry, while the second one is nonnegative since $x \leq \alpha \leq y$ (note that in the second integral the inequality between the limits $\alpha \leq \alpha + \beta - x$ holds since $x \leq \alpha \leq \beta$). Q.E.D.

Notation. Let $n \geq 2$ be a positive integer. Let $\boldsymbol{X} = (X_k)_{1\leq k\leq n}$ be a random vector valued in $\mathbb{R}_+^n$ with independent components. For any $i \leq j$ denote by $\boldsymbol{X}_{i:j}$ the random vector $\boldsymbol{X}_{i:j} := (X_k)_{i\leq k\leq j}$.

Let $\boldsymbol{a} = (a_k)_{1\leq k\leq n} \in \mathbb{R}_+^n$ and let

$$P_{\boldsymbol{X}}(\boldsymbol{a}) = P(X_1 \leq a_1, X_1 + X_2 \leq a_1 + a_2, ..., X_1 + X_2 + ... + X_n \leq a_1 + a_2 + ... + a_n). \tag{1}$$

We also denote

$$S_k = X_1 + ... + X_k,\ s_k = a_1 + ... + a_k,\ \forall k \geq 1$$

so that we can rewrite (1) as $P_{\boldsymbol{X}}(\boldsymbol{a}) = P(S_1 \leq s_1, S_2 \leq s_2, ..., S_n \leq s_n)$.

If $\boldsymbol{Y}$ is another random vector, we denote

$$P_{\boldsymbol{X}}(\boldsymbol{a}|\boldsymbol{Y}) = P(X_1 \leq a_1, X_1 + X_2 \leq a_1 + a_2, ..., X_1 + X_2 + ... + X_n \leq a_1 + a_2 + ... + a_n | \boldsymbol{Y}).$$

With this notation the following result holds:

LEMMA 2.2. *Let* $\boldsymbol{X}$ *be a random vector valued in* $\mathbb{R}_+^n$ *with independent components and* $\boldsymbol{a} \in \mathbb{R}_+^n$. *Suppose that* $X_i \leq_{like} X_{i+1}$ *for some* $1 \leq i \leq n-1$. *Let*

$$X^* = \begin{cases} (X_1, .., X_{i-1}, X_{i+1}, X_i, X_{i+2}, .., X_n), & 2 \leq i \leq n-2 \\ (X_2, X_1, X_3, ..., X_n), & i = 1 \\ (X_1, .., X_{n-2}, X_n, X_{n-1}), & i = n-1 \end{cases}. \tag{2}$$

Then $P_{\boldsymbol{X}}(\boldsymbol{a}) \geq P_{\boldsymbol{X}^*}(\boldsymbol{a})$ *or, equivalently,* $\Delta_i(\boldsymbol{a}) = P_{\boldsymbol{X}}(\boldsymbol{a}) - P_{\boldsymbol{X}^*}(\boldsymbol{a}) \geq 0$.

Proof. Let f_i, f_{i+1} be the densities of X_i and X_{i+1} respectively. Let

$$\delta_i(x,y) = f_i(x) f_{i+1}(y) - f_i(y) f_{i+1}(x). \tag{3}$$

As $X_i \leq_{like} X_{i+1}$, it holds that $0 \leq x \leq y \Rightarrow \delta_i(x,y) \geq 0$ and , moreover, it is obvious that

$$\delta_i(x,y) + \delta_i(y,x) = 0 \tag{4}$$

for all $x, y \geq 0$. We start with $i = 1$ and we note that

$$\begin{aligned} P_{\boldsymbol{X}}(\boldsymbol{a}) &= E\left[P(S_1 \leq s_1, S_2 \leq s_2, ..., S_n \leq s_n | X_1, X_2)\right] = \\ &= E\left[1_{\{X_1 \leq a_1, X_1 + X_2 \leq a_1 + a_2\}} P(X_3 \leq s_3 - X_1 - X_2, .., X_3 + ... + X_n \leq s_n - X_1 - X_2 | X_1, X_2)\right] = \\ &= E\left[1_{\{X_1 \leq a_1, X_1 + X_2 \leq a_1 + a_2\}} P_{\boldsymbol{X}_{3:n}}(s_3 - X_1 - X_2, a_4, .., a_n | \boldsymbol{X}_{1:2})\right] = \\ &= \int_0^{a_1}\int_0^{a_1+a_2-x_1} f_1(x_1) f_2(x_2) P_{\boldsymbol{X}_{3:n}}(s_3 - x_1 - x_2, a_4, .., a_n)\mathrm{d}x_2\mathrm{d}x_1. \end{aligned}$$

Similarly, we get $P_{\boldsymbol{X}^*}(\boldsymbol{a}) = \int_0^{a_1} \int_0^{a_1+a_2-x_1} f_1(x_2) f_2(x_1) P_{\boldsymbol{X}_{3:n}}(s_3 - x_1 - x_2, a_4, .., a_n) \mathrm{d}x_2 \mathrm{d}x_1$.

Thus

$$\Delta_1(\boldsymbol{a}) = \int_0^{a_1} \int_0^{a_1+a_2-x_1} \delta_1(x_1, x_2) P_{\boldsymbol{X}_{3:n}}(s_3 - x_1 - x_2, a_4, .., a_n) \mathrm{d}x_2 \mathrm{d}x_1 = D_{a_1,a_2}(h_1), \tag{5}$$

where $D_{a_1,a_2}(h)$ is defined in Lemma 2.1. and $h_1(x,y) = \delta_1(x,y) P_{\boldsymbol{X}_{3:n}}(s_3 - x_1 - x_2, a_4, .., a_n)$. Applying Lemma 2.1. yields $\Delta_1(\boldsymbol{a}) \geq 0$. The same reasoning holds in general.

For an arbitrary $2 \leq i \leq n-2$, we have

$$P_{\boldsymbol{X}}(\boldsymbol{a}) = E\left[P\left(S_1 \leq s_1, S_2 \leq s_2, ..., S_n \leq s_n \middle| X_{1:(i+1)}\right)\right] =$$
$$= E\left[1_{\{S_1 \leq s_1, S_2 \leq s_2, .., S_{i+1} \leq s_{i+1}\}} P\left(X_{i+2} \leq s_{i+2} - S_{i+1}, .., X_{i+2} + ... + X_n \leq s_n - S_{i+1} \middle| X_{1:(i+1)}\right)\right] =$$
$$= E\left[1_{\{S_1 \leq s_1, S_2 \leq s_2, .., S_{i+1} \leq s_{i+1}\}} P_{\boldsymbol{X}_{(i+2):n}}\left(s_{i+2} - S_{i+1}, a_{i+3}, .., a_n \middle| X_{1:(i+1)}\right)\right].$$

On the other hand,

$$P_{\boldsymbol{X}^*}(\boldsymbol{a}) = E\left[1_{\{S_1 \leq s_1, ..., S_{i-1} \leq s_{i-1}, S_{i-1} + X_{i+1} \leq s_i, S_{i+1} \leq s_{i+1}\}} P\left(X_{i+2} \leq s_{i+2} - S_{i+1}, .., X_{i+2} + ... + X_n \leq s_n - S_{i+1} \middle| X_{1:(i+1)}\right)\right]$$
$$= E\left[1_{\{S_1 \leq s_1, ..., S_{i-1} \leq s_{i-1}, S_{i-1} + X_{i+1} \leq s_i, S_{i+1} \leq s_{i+1}\}} P_{\boldsymbol{X}_{(i+2):n}}\left(s_{i+2} - S_{i+1}, a_{i+3}, ..., a_n \middle| X_{1:(i+1)}\right)\right]$$

and

$$\Delta_i(\boldsymbol{a}) = P_{\boldsymbol{X}}(\boldsymbol{a}) - P_{\boldsymbol{X}^*}(\boldsymbol{a}) =$$
$$= E\left[\left(1_{\{S_1 \leq s_1, S_2 \leq s_2, ..., S_{i+1} \leq s_{i+1}\}} - 1_{\{S_1 \leq s_1, ..., S_{i-1} \leq s_{i-1}, S_{i-1} + X_{i+1} \leq s_i, S_{i+1} \leq s_{i+1}\}}\right) P_{\boldsymbol{X}_{(i+2):n}}\left(s_{i+2} - S_{i+1}, a_{i+3}, ..., a_n \middle| X_{1:(i+1)}\right)\right] =$$
$$= \int_0^{s_1} \int_0^{s_2 - x_1} \cdots \int_0^{s_{i-1} - \sum_{j=1}^{i-2} x_j} \left(\prod_{j=1}^{i-1} f_j(x_j)\right) \int_0^{s_i - \sum_{j=1}^{i-1} x_j} \int_0^{s_{i+1} - \sum_{j=1}^{i} x_j} \delta_i(x_i, x_{i+1}) P_{\boldsymbol{X}_{(i+2):n}}\left(s_{i+2} - \sum_{j=1}^{i+1} x_j, a_{i+3}, .., a_n\right) \mathrm{d}x_{i+1} \mathrm{d}x_i .. \mathrm{d}x_1$$

Hence we obtain

$$\Delta_i(\boldsymbol{a}) = P_{\boldsymbol{X}}(\boldsymbol{a}) - P_{\boldsymbol{X}^*}(\boldsymbol{a}) = \int_0^{s_1} \int_0^{s_2 - x_1} \cdots \int_0^{s_{i-1} - \sum_{j=1}^{i-2} x_j} \left(\prod_{j=1}^{i-1} f_j(x_j)\right) D_{\left(s_i - \sum_{j=1}^{i-1} x_j\right), a_{i+1}}(h_i) \mathrm{d}x_{i-1} \mathrm{d}x_i .. \mathrm{d}x_1,$$

with $h_i(x,y) = \delta_i(x,y) P_{\boldsymbol{X}_{(i+2):n}}\left(s_{i+2} - \sum_{j=1}^{i-1} x_j - x - y, a_{i+3}, ..., a_n\right)$.

According to Lemma 2.1. $D_{\left(s_i - \sum_{j=1}^{i-1} x_j\right), a_{i+1}}(h_i) \geq 0$ from where $\Delta_i(\boldsymbol{a}) \geq 0$.

Finally, the case $i = n-1$ results in a similar way and the proof is completed.

3. MAIN RESULTS

PROPOSITION 3.1. *Let* $\boldsymbol{X} = (X_k)_{1 \leq k \leq n}, \boldsymbol{Y} = (Y_k)_{1 \leq k \leq n}$, *be independent random vectors valued in* $\mathbb{R}_+^n$ *with independent components. Assume that the components of* $\boldsymbol{Y}$ *are identically distributed, let* $\xi_k = X_k - Y_k$ *and let* $L(\xi) = \max(0, \xi_1, \xi_1 + \xi_2, ..., \xi_1 + \xi_2 + .. + \xi_n)$. *Let i* ∈ {1,2,…,*n*-1} *be fixed and*

$$\xi^* = \begin{cases} \left(\xi_1,..,\xi_{i-1},\xi_{i+1},\xi_i,\xi_{i+2},..,\xi_n\right), & 2 \le i \le n-2 \\ \left(\xi_2,\xi_1,\xi_3,...,\xi_n\right), & i=1 \\ \left(\xi_1,..,\xi_{n-2},\xi_n,\xi_{n-1}\right), & i=n-1 \end{cases}. \tag{6}$$

Suppose that $X_i \le_{like} X_{i+1}$.

Then $L(\xi) \le_{st} L(\xi^*)$.

Proof. We have to prove that $P\left(L(\xi)\le t\right) \ge P\left(L(\xi^*)\le t\right)$ for all $t \ge 0$, or, equivalently, that

$P\left(X_1 \le Y_1 + t,...,X_1+...+X_n \le Y_1+...+Y_n+t\right) \ge P\left(X_1^* \le Y_1^* + t,...,X_1^*+...+X_n^* \le Y_1^*+...+Y_n^*+t\right)$,

where X^* is defined in (2) and Y^* is a random vector independent of X^* and having the same distribution as $\boldsymbol{Y}$. Let H be the common distribution of the Y_ks. Then, taking into account the fact that the measure H^n is invariant in respect to permutations, we see that

$$P\left(X_1 \le Y_1 + t,...,X_1+...+X_n \le Y_1+...+Y_n+t\right) = \int_{\mathbb{R}_+^n} P_{\boldsymbol{X}}\left(a_1+t,a_2,..,a_n\right)\mathrm{d}H^n\left(\boldsymbol{a}\right),$$

$$P\left(X_1^* \le Y_1^* + t,...,X_1^*+...+X_n^* \le Y_1^*+...+Y_n^*+t\right) = \int_{\mathbb{R}_+^n} P_{\boldsymbol{X}^*}\left(a_1+t,a_2,..,a_n\right)\mathrm{d}H^n\left(\boldsymbol{a}\right),$$

hence the difference between the two integrals is $\int_{\mathbb{R}_+^n} \Delta_i\left(a_1+t,a_2,...,a_n\right)\mathrm{d}H^n\left(\boldsymbol{a}\right) \ge 0$ according to lemma 2.2.

This completes the proof. QED.

Let now σ be a permutation of {1,2,…,n} and let σ(ξ) be the vector with the components $\left(\xi_{\sigma(k)}\right)_{1\le k\le n}$. Assuming that, in an insurance context, ξ_k represents the loss between two consecutive claims (i.e. claims number k–1 and k), we let

$$L_\sigma = \max\left(0,\xi_{\sigma(1)},\xi_{\sigma(1)}+\xi_{\sigma(2)},...,\xi_{\sigma(1)}+\xi_{\sigma(2)}+..+\xi_{\sigma(n)}\right)$$

be the maximum aggregate loss given by σ(ξ). It is known that $\Psi_\sigma(t) = P\left(L_\sigma > t\right)$ defines the ruin probability at or before the n^{th} claim (see, for instance [1]) . Let e be the identical permutation and ϖ be the inverse permutation, i.e. $\varpi = \begin{pmatrix} 1 & 2 & ... & n \\ n & n-1 & ... & 1 \end{pmatrix}$.

Then the following result holds.

PROPOSITION 3.2. *Let* $\boldsymbol{X} = \left(X_k\right)_{1\le k\le n}, \boldsymbol{Y} = \left(Y_k\right)_{1\le k\le n}$, *be independent random vectors valued in* $\mathbb{R}_+^n$ *with independent components, representing the claim sizes and, respectively, the inter-claim revenues of an insurance surplus process. Assume that the components of* $\mathbf{Y}$ *are identically distributed and that* $X_1 \le_{like} X_2 \le_{like} ... \le_{like} X_n$. *Then* $\Psi_e \le \Psi_\sigma \le \Psi_\varpi$.

Proof. We have to prove that $L_e \le_{st} L_\sigma \le_{st} L_\varpi$ for any permutation σ. Suppose that $\sigma \ne e, \sigma \ne \varpi$ and let $\boldsymbol{X}^* = \sigma\left(\boldsymbol{X}\right)$. Then there exist some $i,j \in \{1,2,..,n-1\}$ such that $\sigma(i) < \sigma(i+1)$ and $\sigma(j) > \sigma(j+1)$. Then $X_{\sigma(i)} \le_{like} X_{\sigma(i+1)} \Leftrightarrow X_i^* \le_{like} X_{i+1}^*$. According to proposition 3.1. $L\left(\xi^*\right) \le_{st} L\left(\tau_i\left(\xi^*\right)\right)$ where τ_i is the transposition between i and i+1. Regarding the index j, according to the same proposition, $L\left(\tau_j\left(\xi^*\right)\right) \le_{st} L\left(\xi^*\right)$, where τ_j is the transposition between j and j+1. In other words, if $\sigma \ne e, \sigma \ne \varpi$, we can both increase and decrease the value of $\Psi_\sigma(t)$.

The only two cases when we cannot do that are when $\sigma = e$ (here we can only increase) and $\sigma = \varpi$ (when we can only decrease). And noting that any permutation can be built from e or ϖ by successively permuting two consecutive numbers, the proof is complete.

Example. In the context of Proposition 3.2., for n = 4, we have the following inequalities chains:

$$\Psi_{1234} \le \Psi_{2134} \le \Psi_{2314} \le \Psi_{2341} \le \Psi_{3241} \le \Psi_{3421} \le \Psi_{4321},$$
$$\Psi_{1234} \le \Psi_{1243} \le \Psi_{1423} \le \Psi_{4123} \le \Psi_{4132} \le \Psi_{4312} \le \Psi_{4321},$$
$$\Psi_{1234} \le \Psi_{1324} \le \Psi_{1342} \le \Psi_{1432} \le \Psi_{4132} \le \Psi_{4312} \le \Psi_{4321}.$$

As a thumb rule, if we can proceed from σ to σ' using inversion of type "if $X_{\sigma(i)} \le_{like} X_{\sigma(i+1)}$, we interchange them", then $\Psi_\sigma \le \Psi_{\sigma'}$.

COROLLARY 3.3. *Let* $\boldsymbol{X} = (X_k)_{1\le k\le n}, \boldsymbol{Y} = (Y_k)_{1\le k\le n}$, *be independent random vectors valued in* $\mathbb{R}_+^n$ *with independent components. Suppose that the components of* **Y** *are identically distributed and the components of* **X** *are* Gamma *distributed:* $X_i \sim \text{Gamma}(\nu_i, \lambda_i)$.

If $\nu_1 \le \nu_2 \le \ldots \le \nu_n$ *and* $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ *then* $\Psi_e \le \Psi_\sigma \le \Psi_\varpi$.

Proof. It is easy to see that $\text{Gamma}(\nu,\lambda) \le_{like} \text{Gamma}(\nu',\lambda')$ if and only if $\nu \le \nu'$ and $\lambda \ge \lambda'$. Indeed, in this case the ratio $\frac{f_1}{f_2}(x) = x^{\nu-\nu'} e^{-x(\lambda-\lambda')}$ is decreasing if and only $\nu \le \nu'$ and $\lambda \ge \lambda'$.

Or, otherwise written $\text{Gamma}(\nu,\lambda) \le_{like} \text{Gamma}(\nu',\lambda')$ iff $\nu \le \nu'$ and $\lambda \ge \lambda'$ and the result is immediate from Proposition 3.2.

ACKNOWLEDGEMENTS

Gheorghiță Zbăganu acknowledges the support from a research grant of the Romanian National Authority for Scientific Research CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0908.

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Received October 11, 2015