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EXACT SOLUTIONS TO THE RESONANT NONLINEAR SCHRÖDINGER EQUATION WITH BOTH SPATIO-TEMPORAL AND INTER-MODAL DISPERSIONS

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Abstract. The resonant nonlinear Schrödinger equation with both spatio-temporal and inter-modal dispersions, which describes the propagation dynamics of optical solitons and Madelung fluids, is studied by analytical techniques. Three types of nonlinearities are considered in this work, namely the Kerr law, dual-power law, and parabolic law. The powerful integration tool is the G'/G expansion method that is used to obtain families of periodic, singular, and rational bright-type soliton solutions, as well as dark-type soliton solutions.

Key words: optical solitons, spatio-temporal dispersion, inter-modal dispersion.

1. INTRODUCTION

It is known that the nonlinear dynamics of optical solitons and Madelung fluids are modeled by a generic resonant nonlinear Schrödinger equation (R-NLSE). Generally, a specific resonant term should be taken into account in the study of chiral solitons in quantum Hall effect [1–6]. Recently, many researchers focused their studies on the construction of exact soliton solutions to several nonlinear partial differential equations, and different tools of integration were adopted, such as the solitary wave ansatz scheme, the Riccati equation expansion method, the Bernoulli equation expansion method, the tanh function expansion method and the first integral approach [1–30]. It should be noted that the governing equation for the propagation of optical solitons in nonlinear media is well-posed only when the additional spatio-temporal dispersion (STD) is considered [16]. Hence, in the presence of inter-modal dispersion (IMD), the well-posed nonlinear dynamical model that will be investigated in this work is given by the following R-NLSE:

$$iq_t + aq_{xx} + bq_{xt} + cF(|q|^2)q + \gamma\left(\frac{|q|_{xx}}{|q|}\right)q = \mathrm{i}\delta q_x, \tag{1}$$

where $q(x,t)$ is the complex wave profile, and x and t are the spatial and temporal variables, respectively. Here, a and b represent the coefficients of group-velocity dispersion and STD, respectively, while c and γ are the coefficients of non-Kerr law nonlinearity and resonant nonlinearity, respectively. Finally δ gives the coefficient of IMD. The first term in Eq. (1) represents the time evolution term.

In this paper, Eq. (1) will be studied analytically by using the G'/G expansion method. Three types of nonlinearities, namely Kerr law, dual-power law, and parabolic law are considered in this work. As a result, exact soliton solutions such as hyperbolic function traveling wave solutions, trigonometric function traveling wave solutions, and rational wave solutions are reported.

2. FAMILIES OF EXACT SOLUTIONS TO THE R-NLSE (1)

In order to construct explicit traveling wave solutions to the R-NLSE (1), we first make the following hypothesis:

$$q(x,t)=\psi(\xi)\exp[\mathrm{i}\phi(x,t)], \tag{2}$$

and

$$\xi = x - vt, \tag{3}$$

$$\phi(x,t) = -\kappa x + \omega t + \theta, \tag{4}$$

where $\psi(\xi)$ represents the shape of the traveling wave, while $\phi(x,t)$ is the phase component of the wave. Here, v, κ, ω, and θ give the velocity, frequency, wave number, and phase constant, respectively.

Substituting Eqs. (2–4) into Eq. (1), we obtain

$$v = \frac{\delta + 2a\kappa - b\omega}{b\kappa - 1}, \tag{5}$$

$$(a - bv + \gamma)\psi'' + (b\kappa\omega - \omega - \delta\kappa - a\kappa^2)\psi + cF(\psi^2)\psi = 0, \tag{6}$$

where the prime denotes differentiation with respect to the variable ξ.

Equation (5) gives the velocity of the traveling wave, while Eq. (6) will be solved in the following subsections by using the G'/G expansion approach. In what follows, three types of nonlinear media will be considered in detail. They are Kerr law, dual-power law, and parabolic law nonlinear media.

2.1. KERR LAW NONLINEARITY

For Kerr law nonlinearity, $F(\psi^2) = \psi^2$, then Eq. (6) reduces to

$$(a - bv + \gamma)\psi'' + (b\kappa\omega - \omega - \delta\kappa - a\kappa^2)\psi + c\psi^3 = 0. \tag{7}$$

Based on the balancing principle, we can assume that the solutions of Eq. (7) have the following form

$$\psi(\xi) = l_0 + l_1\left(\frac{G'(\xi)}{G(\xi)}\right), \tag{8}$$

where l_0 and l_1 are real constants to be determined later, and $G(\xi)$ satisfies

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0. \tag{9}$$

Inserting Eqs. (8) and (9) into Eq. (7), and then equating the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^m$, where $m = 0, 1, 2, 3$, to zero yields a system of nonlinear algebraic equations. Solving this system and using Eq. (6), we get

$$l_0 = \pm\frac{\lambda}{2}\sqrt{\frac{2[b(\delta + 2a\kappa - b\omega) - (a+\gamma)(b\kappa - 1)]}{c(b\kappa - 1)}}, \tag{10}$$

$$l_1 = \pm\sqrt{\frac{2[b(\delta + 2a\kappa - b\omega) - (a+\gamma)(b\kappa - 1)]}{c(b\kappa - 1)}}, \tag{11}$$

$$\omega = \frac{(\lambda^2 - 4\mu)b(\delta + 2a\kappa) - (\lambda^2 - 4\mu)(a+\gamma)(b\kappa - 1) - 2(b\kappa - 1)\kappa(\delta + a\kappa)}{(\lambda^2 - 4\mu)b^2 - 2(b\kappa - 1)^2}, \tag{12}$$

where λ, μ, κ, and θ are arbitrary constants.

Equation (12) gives the wave number and the corresponding constraint condition

$$(\lambda^2 - 4\mu)b^2 \neq 2(b\kappa - 1)^2 . \tag{13}$$

Additionally, Eqs. (10) and (11) introduce the restriction

$$c(b\kappa - 1)[b(\delta + 2a\kappa - b\omega) - (a+\gamma)(b\kappa - 1)] > 0 . \tag{14}$$

Finally, using the solutions of Eq. (6), we can obtain families of exact *traveling wave solutions* of R-NLSE (1) with Kerr law nonlinearity, which are listed as follows:

Case 1. When $\lambda^2 - 4\mu > 0$, Eq. (1) with Kerr law nonlinearity admits explicit traveling wave solution in terms of hyperbolic functions

$$q(x,t) = \pm\sqrt{\frac{(\lambda^2 - 4\mu)[b(\delta + 2a\kappa - b\omega) - (a+\gamma)(b\kappa - 1)]}{2c(b\kappa - 1)}} \times \left\{ \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right)} \right\} e^{i(-\kappa x + \omega t + \theta)} , \tag{15}$$

where C_1 and C_2 are arbitrary constants, $\xi = x - \dfrac{\delta + 2a\kappa - b\omega}{b\kappa - 1}t$ and ω is given by Eq. (12).

It must be noted that if we take $C_1 = 0$ and $C_2 \neq 0$, the solution (15) degenerates to the *singular soliton solution*

$$q(x,t) = \pm\sqrt{\frac{(\lambda^2 - 4\mu)[b(\delta + 2a\kappa - b\omega) - (a+\gamma)(b\kappa - 1)]}{2c(b\kappa - 1)}} \coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) e^{i(-\kappa x + \omega t + \theta)} , \tag{16}$$

and if we take $C_2 = 0$ and $C_1 \neq 0$, the solution (15) becomes to *dark soliton solution*

$$q(x,t) = \pm\sqrt{\frac{(\lambda^2 - 4\mu)[b(\delta + 2a\kappa - b\omega) - (a+\gamma)(b\kappa - 1)]}{2c(b\kappa - 1)}} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right) e^{i(-\kappa x + \omega t + \theta)} . \tag{17}$$

Case 2. When $\lambda^2 - 4\mu < 0$, Eq. (1) with Kerr law nonlinearity admits analytical traveling wave solution in terms of trigonometric functions

$$q(x,t) = \pm\sqrt{\frac{(4\mu - \lambda^2)[b(\delta + 2a\kappa - b\omega) - (a+\gamma)(b\kappa - 1)]}{2c(b\kappa - 1)}} \times \left\{ \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right)} \right\} e^{i(-\kappa x + \omega t + \theta)} , \tag{18}$$

where C_1 and C_2 are arbitrary constants, $\xi = x - \dfrac{\delta + 2a\kappa - b\omega}{b\kappa - 1}t$ and ω is given by Eq. (12).

It must be noted that if we take $C_1 = 0$ and $C_2 \neq 0$, the trigonometric function traveling wave solution (18) degenerates to a *singular periodic solution*

$$q(x,t)=\pm\sqrt{\frac{(4\mu-\lambda^2)[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}{2c(b\kappa-1)}}\ \cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\ e^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{19}$$

and if we take $C_2=0$ and $C_1\neq 0$, the solution (15) becomes another singular periodic solution

$$q(x,t)=\pm\sqrt{\frac{(4\mu-\lambda^2)[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}{2c(b\kappa-1)}}\ \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)\ e^{\mathrm{i}(-\kappa x+\omega t+\theta)}. \tag{20}$$

Case 3. When $\lambda^2-4\mu=0$, Eq. (1) with Kerr law nonlinearity admits analytical traveling wave solution in terms of *rational functions*

$$q(x,t)=\pm\sqrt{\frac{2[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}{c(b\kappa-1)}}\left(\frac{C_2}{C_1+C_2\left(x-\frac{\delta+2a\kappa-b\omega}{b\kappa-1}t\right)}\right)e^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{21}$$

where C_1 and C_2 are arbitrary constants, and ω is given by Eq. (12).

Remark. Eq. (21) is also known as the plane wave solution. Additionally, we can also use Jacobi elliptic equation expansion method, Riccati equation expansion approach, and ansatz scheme to construct exact solutions of Eq. (7); these methods were recently applied in Refs. [17, 18].

2.2. DUAL-POWER LAW NONLINEARITY

For dual-power law nonlinearity, $F(\psi^2)=\psi^{2n}+\alpha\psi^{4n}$, then Eq. (6) reduces to

$$(a-bv+\gamma)\psi''+(b\kappa\omega-\omega-\delta\kappa-a\kappa^2)\psi+c\psi^{2n+1}+c\alpha\psi^{4n+1}=0, \tag{22}$$

where α is a constant, and $0<n<2$.

According to the balancing principle, we first make the following assumption:

$$\psi(\xi)=l_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{\frac{1}{2n}}, \tag{23}$$

where l_1 is a real constant to be determined later.

Putting Eqs. (9) and (23) into Eq. (22), and equating the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^m$, where $m=\frac{1}{2n}-2,\frac{1}{2n}-1,\frac{1}{2n},\frac{1}{2n}+1,\frac{1}{2n}+2$, to zero yields a system of nonlinear algebraic equations. Solving this system and using Eq. (6), we obtain

$$l_1=\pm\left\{\frac{(1+2n)[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}{4c\alpha n^2(b\kappa-1)}\right\}^{\frac{1}{4n}}, \tag{24}$$

$$\lambda=\frac{cn}{1+n}\left[\frac{(1+2n)(b\kappa-1)}{c\alpha[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}\right]^{\frac{1}{2}}, \tag{25}$$

$$\mu = 0, \tag{26}$$

$$\omega = \frac{1}{b\kappa - 1}\left[\frac{c(1+2n)}{4\alpha(1+n)^2} + \delta\kappa + a\kappa^2\right], \tag{27}$$

where κ and θ are arbitrary constants.

Equation (27) gives the corresponding wave number that possesses the restriction

$$b\kappa \neq 1. \tag{28}$$

Additionally, Eqs. (24) and (25) introduce the constraint condition

$$c\alpha(b\kappa - 1)(1+2n)[b(\delta + 2a\kappa - b\omega) - (a+\gamma)(b\kappa - 1)] > 0. \tag{29}$$

It should be noted that when $\mu = 0$, the solution of Eq. (9) is given by

$$G(\xi) = C_1 + C_2 \exp(-\lambda\xi), \tag{30}$$

where C_1 and C_2 are arbitrary constants.

Finally, explicit traveling wave solution of Eq. (1) with dual-power law nonlinearity is obtained, which is given by

$$q(x,t) = \pm\left(-\frac{1+2n}{2\alpha(1+n)}\frac{C_2 e^{-\frac{cn}{1+n}\left[\frac{(1+2n)(b\kappa-1)}{c\alpha[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}\right]^{\frac{1}{2}}\xi}}{C_1 + C_2 e^{-\frac{cn}{1+n}\left[\frac{(1+2n)(b\kappa-1)}{c\alpha[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}\right]^{\frac{1}{2}}\xi}}\right)^{\frac{1}{2n}} e^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{31}$$

where $\xi = x - \dfrac{\delta + 2a\kappa - b\omega}{b\kappa - 1}t$ and ω is given by Eq. (27).

It must be noted that if we take $C_1 = C_2$, the traveling wave solution (31) reduces to the *topological (dark) soliton*

$$q(x,t) = \pm\left\{-\frac{1+2n}{\alpha(1+n)}\left(1 - \tanh\left(-\frac{cn}{2(1+n)}\left[\frac{(1+2n)(b\kappa-1)}{c\alpha[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}\right]^{\frac{1}{2}}\xi\right)\right)\right\}^{\frac{1}{2n}} e^{\mathrm{i}(-\kappa x+\omega t+\theta)}, \tag{32}$$

and if we take $C_1 = -C_2$, the traveling wave solution (31) reduces to the *singular soliton*

$$q(x,t) = \pm\left\{-\frac{1+2n}{\alpha(1+n)}\left(1 - \coth\left(-\frac{cn}{2(1+n)}\left[\frac{(1+2n)(b\kappa-1)}{c\alpha[b(\delta+2a\kappa-b\omega)-(a+\gamma)(b\kappa-1)]}\right]^{\frac{1}{2}}\xi\right)\right)\right\}^{\frac{1}{2n}} e^{\mathrm{i}(-\kappa x+\omega t+\theta)}. \tag{33}$$

Remark. We can also use the Lie group method and non-auto-Bäcklund transformation [19, 20] to construct analytical solutions of Eq. (22). Additionally, it should be noted that when $n = 1$, the dual-power law nonlinearity will reduce to parabolic law nonlinearity. Therefore, exact traveling wave solutions of Eq. (1) with parabolic law nonlinearity can be obtained, which are given by Eqs. (31–33) in which $n = 1$.

3. CONCLUSION

This work presented a detailed analytical study of a generic resonant nonlinear Schrödinger equation with both spatio-temporal and inter-modal dispersions. Three kinds of nonlinear media have been considered in this paper, namely the Kerr law, dual-power law, and parabolic law nonlinear media. Exact traveling wave solutions in terms of hyperbolic, trigonometric, and rational functions were constructed via the G'/G expansion approach. According to the different values of two arbitrary constants C_1 and C_2, families of periodic, singular, and rational bright-type soliton solutions, as well as dark-type soliton solutions were derived along with the corresponding existence conditions for such exact solutions.

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