PERIODIC SOLUTIONS OF DAMPED DUFFING-TYPE EQUATIONS WITH SINGULARITY

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Abstract. We consider a second order equation of Duffing type. By applying Mawhin’s continuation theorem and a relationship between the periodic and the Dirichlet boundary value problems for second order ordinary differential equations, we prove that the given equation has at least one positive periodic solution when the singular forces exhibits certain some strong force condition near the origin and with some semilinear growth near infinity. Recent results in the literature are generalized and complemented.

Key words: periodic solution, singular equations, Mawhin’s continuation theorem.

1. INTRODUCTION

This paper is devoted to the existence of positive $T$-periodic solutions for the Duffing-type equation

$$x'' + Cx' + g(t, x) = 0,$$

where $C \in \mathbb{R}$ is a constant, $g \in C(R \times (0, +\infty), \mathbb{R})$ is $T$-periodic with the first variable, and exhibits a repulsive singularity at $x = 0$. Following the notion in [6], we say that (1.1) has a repulsive singularity at $x = 0$ if

$$\lim_{x \to 0^+} g(t, x) = -\infty$$

uniformly in $t \in \mathbb{R}$, whereas (1.1) has an attractive singularity at $x = 0$ if

$$\lim_{x \to 0^+} g(t, x) = +\infty$$

uniformly in $t \in \mathbb{R}$.

As is well known that the Duffing equation named after the German electrical engineer Georg Duffing in 1918, has been widely used in physics, economics, engineering, and many other physical phenomena. An important question is whether this equation can support periodic solutions. Hence, it has been extensively investigated by numerous researchers in recent years. See for example [10, 13, 14, 19] and the references therein. According to the growth speed of $g$, (1.1) can be classified into the following three cases, (S$_1$) Superlinear case: $g(t, s) / s \to +\infty$, as $|s| \to +\infty$; (S$_2$) Superlinear case: $0 < k \leq g(t, s) / s \leq K < +\infty$, as $|s| \to +\infty$; (S$_3$) Superlinear case: $g(t, s) / s \to 0$, as $|s| \to +\infty$.

When $C = 0$, Eq. (1.1) becomes

$$x'' + g(t, x) = 0 \quad ; \quad x > 0$$

we recall the following results. Let $g(t, x) = g(x) - e(t)$, where $g \in C(R_+, \mathbb{R})$ and $e \in C(R, \mathbb{R})$ is $T$-periodic satisfies the following strong force condition at $x = 0$,

$$\lim_{x \to 0^+} g(x) = -\infty \quad \text{and} \quad \lim_{x \to 0^+} \int_0^x g(s) ds = +\infty$$
with \( g \) is superlinear at \( x \to +\infty \), \( \lim_{x \to +\infty} \frac{g(x)}{x} = +\infty \), Fonda, Manásevich, and Zanolin [5] used the Poincaré - Birkhoff theorem to obtain the existence of positive periodic solutions, including all subharmonics. Similarly, when \( g(t,x) \) is superlinear at \( x \to +\infty \) and satisfies the following strong force condition at \( x = 0 \): there are positive constants \( c, c', \nu \) such that \( \nu \geq 1 \) and

\[
 cx^{-\nu} \leq -g(t,x) \leq c'x^{-\nu}
\]

for all \( t \) and all \( x \) sufficiently small, del Pino and Manásevich proved in [2] the existence of infinitely many periodic solutions to (1.2).

When \( g(t,x) \) is semilinear at \( x \to +\infty \), del Pino, Manásevich, and Montero [3] proved the existence of at least one positive \( T \)-periodic solution of (1.2) if \( g(t,x) \) satisfies (1.3) near \( x = 0 \), and the following nonresonance conditions at \( x = +\infty \): there is an integer \( k \geq 0 \) and a small constant \( \varepsilon > 0 \) such that

\[
 \left( \frac{k\pi}{T} \right)^2 + \varepsilon \leq \frac{g(t,x)}{x} \leq \left( \frac{(k+1)\pi}{T} \right)^2 - \varepsilon
\]

for all \( t \) and all \( x \gg 1 \).

The generalization of [3] was done for equation of the type of \( \dot{x}^n + g(x) = e(t) \) in [14]. Assume that \( g(x) \) satisfies

\[
 \lim_{x \to 0^+} g(x) = -\infty , \quad \lim_{x \to 0^-} G(x) = +\infty , \quad (G(x) = \int_0^x g(s)ds)
\]

and

\[
 \left( \frac{k\pi}{T} \right)^2 + \varepsilon \leq \frac{g(x)}{x} \leq \left( \frac{(k+1)\pi}{T} \right)^2 - \varepsilon ,
\]

for all \( t \) and all \( x \gg 1 \). Wang [14] used the phase-plane analysis methods proved that Eq.(1.1) has at least one positive \( T \)-periodic solution.

We note that conditions (1.4) and (1.5) are the standard uniform nonresonance conditions with respect to the Dirichlet boundary condition, not with respect to the periodic boundary condition. For example, \( x'' + \lambda x = x^{-3} + h(t) \), where \( \lambda > 0 \) and \( h \in C(R,R) \) is \( 2\pi \)-periodic. Nonresonance holds when \( \lambda \neq (k/2)^2 \), \( k = 1,2,\ldots, \) i.e., \( \lambda \) is not an eigenvalue of the Dirichlet problem.

Some classical tools have been used in the literature to study singular equations. These classical tools include the degree theory [7, 15, 18], the method of upper and lower solutions [12], Schauder’s fixed point theorem [1], some fixed point theorems in cones for completely continuous operators [8, 16, 17]. It seems the periodic boundary value problem for singular differential equations is closely related to the Dirichlet boundary value problem. A relationship between periodic and Dirichlet boundary value problems for second-order differential equations with singularities is establish in [21]. As mentioned above, this paper is mainly motivated by the recent papers [3, 14]. The result is obtained using Mawhin’s continuation theorem and such a relationship between the periodic boundary value problem and the Dirichlet boundary value problem, thanks to a priori estimates on the solutions of a suitable family of problems. Compared with [3,14], the main novelty in the paper is represented by the conditions at infinity, which remind of a situation between the first and the second eigenvalue, but are more general.

Now we present our main result.

**THEOREM 1.1.** Let the following assumptions hold.

\((H_1)\) There exist a constant \( R_0 > 0 \) and a function \( g_0 \in C((0,\infty),R) \) such that \( g(t,x) \leq -g_0(x) \) for all \( t \) and all \( 0 < x \leq R_0 \), where \( g_0 \) satisfies the strong force condition, i.e.

\[
 \lim_{x \to 0^+} g_0(x) = +\infty \quad \text{and} \quad \lim_{x \to 0^+} \int_0^x g_0(s)ds = -\infty .
\]
There exist $T$-periodic continuous functions $a, b$ such that
\[
 a(t) \leq \liminf_{x \to +\infty} \frac{g(t,x)}{x} \leq \limsup_{x \to +\infty} \frac{g(t,x)}{x} \leq b(t)
\]
uniformly in $t \in [0, T]$. Moreover,
\[
 \overline{a} > 0 \quad \text{and} \quad \underline{\lambda}(b) > 0
\]
where \( \overline{a} = \frac{1}{T} \int_0^T a(t) \, dt \) and \( \{ \underline{\lambda}(q) \} \) denotes the first anti-periodic eigenvalues of
\[
x'' + (\lambda + q(t))x = 0
\]
subject to the anti-periodic boundary conditions.

Then Eq. (1.1) has at least one positive $T$-periodic solution.

Note that when $q \equiv 0$ and $\underline{\lambda}(0) = 0$ and $\lambda_k(0) = k\pi / T$ for all $k \in \mathbb{N}$. These eigenvalues coincide with the constants in conditions (1.4) and (1.5). The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, by the use of Mawhin’s continuation theorem, we will prove the main results.

2. PRELIMINARIES

Let us first introduce some known results on eigenvalues. Let $q(t)$ be a $T$-periodic potential such that $q \in L^1(\mathbb{R})$. Consider the eigenvalue problems of (1.8) subject to the $T$-periodic boundary condition
\[
x(0) - x(T) = x'(0) - x'(T) = 0
\]
or to the anti-$T$-periodic boundary condition
\[
x(0) + x(T) = x'(0) + x'(T) = 0.
\]
Denote by $\lambda_1^D(q) < \lambda_2^D(q) < \cdots < \lambda_n^D(q) < \cdots$ all eigenvalues of (1.8) with the Dirichlet boundary condition
\[
x(0) = x(T) = 0.
\]

The following are the standard results for eigenvalues. See, e.g Reference [9, 23]. A partial generalization of these results to the one-dimensional $p$-Laplacian with potentials is given in Reference [22].

**Lemma 2.1.** There exist two sequences $\{ \underline{\lambda}_n(q) : n \in \mathbb{N} \}$ and $\{ \overline{\lambda}_n(q) : n \in \mathbb{Z}^+ \}$ such that:

( $E_1$) [9, Theorem 2.1, see also 22, Theorem A)]. \( \lambda \) is an eigenvalue of (1.8–2.1) (or (1.8–2.2)) if and only if $\lambda = \underline{\lambda}_n(q)$ for some even (or odd) integer $n$;

( $E_2$) [9, Theorem 2.1, see also 22, Theorem A]).
\[
-\infty < \underline{\lambda}_1(q) < \underline{\lambda}_2(q) \leq \underline{\lambda}_3(q) < \cdots < \underline{\lambda}_n(q) \leq \cdots < \lambda_1(q) < \cdots < \lambda_2(q) \leq \cdots < \lambda_n(q) < \cdots \quad \text{and} \quad \underline{\lambda}_n(q) \to +\infty,
\]
\[
\overline{\lambda}_n(q) \to +\infty \quad \text{as} \quad n \to +\infty;
\]

( $E_3$) [22, Remark 4.3]. The comparison results hold for all of these eigenvalues. If $q_1 \leq q_2$ then $\lambda_n^D(q_1) \geq \lambda_n^D(q_2)$, $\lambda_n(q_1) \geq \lambda_n(q_2)$, $\lambda_n^D(q_1) \geq \lambda_n^D(q_2)$ for any $n \geq 1$;

( $E_4$) [22, Theorem 4.3]. For any $n \geq 1$,
\[
\underline{\lambda}_n(q) = \min \{ \lambda_n^D(q) : t_0 \in \mathbb{R} \}, \quad \overline{\lambda}_n(q) = \max \{ \lambda_n^D(q) : t_0 \in \mathbb{R} \},
\]
where \( q_{s_0}(t) \) denotes the translation of \( q(t) : q_{s_0}(t) \equiv q(t + t_0) \);

\( (E_s) \) [23, Theorem 5.2]. \( \lambda_n(q), \lambda_{n-s}(q), \) and \( \lambda_{n-s}^D(q) \) continuous in \( q \) in the \( L^1 \)-topology of \( L'(0, T) \);

\( (E_s) \) (Variational characterization). The first eigenvalue \( \lambda_1^D \) can be found as

\[
\lambda_1^D(q) = \inf_{x \in H, \|x\| = 1} \int_0^T (x^2(t) - q(t)x^2(t))dt / \int_0^T x^2(t)dt.
\]

In particular, \( \lambda_1^D(q) \leq -\bar{q} \), where \( \bar{q} \) is the mean value.

As we know, continuation theorems play an important role in studying the existence of periodic solutions of the second order differential equations. We now introduce theorem given by Mawhin [11].

**Definition 2.1.** Let \( : \text{dom} L \to Z \) be a linear operator, and \( X, Z \) be real Banach space. \( L \) is said to be a Fredholm operator of index zero provided that

(i) \( \text{Im} L \) is a closed subset of \( Z \),

(ii) \( \dim \text{Ker} L = \text{codim} \text{Im} L < +\infty \).

Set \( X = \text{Ker} L \oplus X_1, \quad Z = Z_0 \oplus \text{Im} L \). We will also need the projector \( P : X \to \text{Ker} L \) and \( Q : Z \to Z_0 \) are continuous such that \( \text{Im} P = \text{Ker} L, \quad \text{Ker} Q = \text{Im} L \). Then it follows that \( L|_{\text{dom} L \setminus \text{Ker} L} : \text{dom} L \setminus \text{Ker} L \to \text{Im} L \) is invertible. We denote the inverse of that map by \( K \) and \( P = KQ \).

**Definition 2.2.** Let \( \lambda \) be a Fredholm operator of index zero and let \( N \) be \( L \)-compact on \( \Omega \). Assume that the following conditions are satisfied:

(i) \( \text{Im} \lambda \lambda x + \tau N x \neq 0 \), for each \( (x, \tau) \in [(\text{dom} \lambda \lambda \setminus \text{Ker} \lambda \lambda) \cap \partial \Omega] \times (0, 1) \);

(ii) \( \text{Im} \lambda \lambda x \neq 0 \) for each \( x \in \text{Ker} \lambda \lambda \cap \partial \Omega \);

(iii) \( D_0(QN \mid_{\text{Ker} \lambda \lambda}, \Omega \cap \text{Ker} \lambda \lambda) \neq 0 \), where \( Q : Z \to Z \) is a continuous projector such that \( \text{Ker} Q = \text{Im} \lambda \lambda \) and \( D_0 \) is the Brouwer degree.

Then the equation \( Lx + Nx = 0 \) has at least one solution in \( \text{dom} \lambda \lambda \cap \partial \Omega \).

We refer the reader to [20] for more details about degree theory.

### 3. PROOF OF THEOREM 1.1

We will apply Lemma 2.2 to the singular problem (1.1). Let \( X = \{x : R \to R \to R \} \) and satisfies \( x(t + T) = x(t) \), endowed with the \( C^n \)-norm, clearly, \( X \) is a Banach space, and let \( Z = L'(0, T; R) \) with the \( L^1 \)-norm. Let \( \text{dom} L = \{x \in X : x' \text{ is absolutely continuous on } R \} \) and \( L : \text{dom} L \to Z \) be the operator defined by \( (Lx)(t) = x'(t) \), and \( N : X \to Z \) by \( (N x)(t) = Cx'(t) + g(t, x(t)) \). Then (1.1) is equivalent to the operator equation \( Lx + Nx = 0 \).

Define projectors \( P : X \to X \) and \( Q : Z \to Z \) by

\[
P x = \frac{1}{T} \int_0^T x(s)ds \quad \text{and} \quad Q z = \frac{1}{T} \int_0^T z(s)ds.
\]

It is easy to see that \( \text{Ker} L = R, \quad \text{Im} L = \{z \in Z : \int_0^T z(t)dt = 0\}, \quad \text{Ker} Q = \text{Im} L, \quad \text{Im} P = \text{Ker} L, \) and then \( L \) is a Fredholm linear mapping with zero index.

Let \( K \) denote the inverse of \( L|_{\text{Ker} \lambda \lambda \mid \text{dom} L} \). Then we have
\[ [Kz](t) = \int_0^T G(t,s)z(s)ds, \tag{3.2} \]

where \( G(t,s) \) is the Green function.

Now we consider the following (homotopy) family of (1.1)
\[ x''(t) + \tau(Cx'(t) + g(t,x(t))) = 0, \quad \tau \in (0,1] \tag{3.3} \]
i.e. the operator equation \( Lx + \tau Nx = 0 \).

**Lemma 3.1.** Under the assumptions as in Theorem 1.1, there exist \( B_1 > B_0 > 0 \) such that for any positive solution \( x(t) \) to (3.3–2.2), there exists some \( t_0 \in [0, T] \) such that
\[ B_0 < x(t_0) < B_1. \tag{3.4} \]

**Proof.** Let \( x(t) \) be a positive solution of (3.3)–(2.2). By condition (H_1), there is \( B_0 > 0 \) such that \( g(t,s) < 0 \) for all \( 0 < s < B_0 \). Integrating (3.3) from 0 to \( T \), we get
\[ \tau \int_0^T g(t,x(t))dt = -\int_0^T x''(t)dt - \tau \int_0^T Cx'dt = 0. \]
Thus \( \int_0^T g(t,x(t))dt = 0 \), there exist \( t^* \in [0, T] \) such that \( x(t^*) > B_0 \).

Next, noticing (1.7), we take some constant \( \varepsilon_0 \in (0, \min\{\bar{a}, \underline{\lambda}(b)\}) \). By condition (H_2) there is \( B_1(B_0) \) large enough such that
\[ a(t) - \varepsilon_0 \leq \frac{g(t,s)}{s} \leq b(t) + \varepsilon_0 \tag{3.5} \]
for all \( t \) and \( s \geq B_1 \). We assert that \( x(t_*) < B_1 \) for some \( t_* \). Once this is proved, combining with the fact \( x(t^*) > B_0 \), we know that (3.4) is necessarily satisfied for some \( t_0 \).

Now we prove the assertion by contradiction. Assume that \( x(t) \geq B_1 \) for all \( t \). Define \( p(t) := \frac{g(t,x(t))}{x(t)} \). By (3.5), \( a(t) - \varepsilon_0 \leq p(t) \leq b(t) + \varepsilon_0 \), for all \( t \). Moreover, \( x(t) \) satisfies the following differential equation \( x'' + \tau(Cx' + p(t)x) = 0 \). Write \( x = \tilde{x} + \bar{x} \), where \( \bar{x} = \frac{1}{T} \int_0^T x(t)dt \), then \( \tilde{x}'(t) = x'(t) \) and \( \tilde{x} \) satisfies
\[ -\tilde{x}'' - \tau C\tilde{x}' = \tau(p(t)\tilde{x} + p(t)\bar{x}). \tag{3.6} \]
Integrating the equation (3.6) from 0 to \( T \), we have
\[ \int_0^T p(t)\tilde{x}(t)dt = -\bar{x}\int_0^T p(t)dt. \tag{3.7} \]
Multiplying (3.6) by \( \tilde{x} \) and integrate, using the fact that \( \int_0^T \tilde{x}'(t)\tilde{x}(t)dt = 0 \), we get
\[ \|\bar{x}\|^2 = \tau \int_0^T p(t)\tilde{x}^2(t)dt + \tau\bar{x}\int_0^T p(t)\tilde{x}(t)dt = \tau \int_0^T p(t)\tilde{x}^2(t)dt - \tau\bar{x}^2(t)\int_0^T p(t)dt \leq \]
\[ \leq \tau\int_0^T p(t)\tilde{x}^2(t)dt \leq \int_0^T p(t)\tilde{x}^2(t)dt, \tag{3.8} \]
where (3.7) is used and the last inequality follows from the fact that \( \int_0^T p(t)dt > T(\bar{a} - \varepsilon_0) > 0 \).
Note that \( \tilde{x}(t_0) = 0 \) for some \( t_0 \), \( \tilde{x}(t_0 + T) = 0 \), so \( \tilde{x}(t) \in H^1_0(t_0, t_0 + T) \). We assert that \( \tilde{x} \equiv 0 \). On the contrary, assume that \( \tilde{x} \neq 0 \). Now by (3.8), the first Dirichlet eigenvalue

\[
\lambda_1^D(p) = \inf_{\varphi \in H^1_0(t_0, t_0 + T) \setminus \{0\}} \frac{\int_{t_0}^{t_0+T} (\varphi'^2(t) + p(t)\varphi^2(t)) dt}{\int_{t_0}^{t_0+T} \varphi^2(t) dt} \leq 0.
\]

So, we conclude that \( \lambda_1(p) = \min \{ \lambda_1^D(p) \} \leq 0 \). However, from \( p(t) < b(t) + \varepsilon_0 \), we have \( \lambda_1(p) \geq \lambda_1(b + \varepsilon_0) = \lambda_1(b) - \varepsilon_0 > 0 \) by the choice of \( \varepsilon_0 \).

We have a contradiction, which shows that \( \tilde{x} = 0 \), thus \( x = \bar{x} \). Now it follows from (3.7) that \( \bar{x} = 0 \) and \( x = 0 \). This contradicts with the fact that \( x \) is a positive solution. We finished the proof of the assertion needed.

**Lemma 3.2.** Assume \( \lambda_1(b) > 0 \) of the equation \( y'' + (\lambda + b(t))y = 0 \), then

\[
\|y''\|_b^2 \geq \int_0^T b(t + t_0)y^2(t) dt + \lambda_1^D(b_{t_0}) \int_0^T y^2(t) dt.
\]

**Proof.** From Lemma 2.1 in (4.4), we have \( \lambda_1^D(b_{t_0}) \geq \lambda_1(b) > 0 \) for all \( t_0 \in R \). Then, by the theory of differential operators [4], the eigenvalues of \( y'' + (\lambda + b(t + t_0))y = 0 \) with Dirichlet boundary conditions \( y(0) = y(T) = 0 \) form a sequence \( \lambda_1^D(b_{t_0}) < \lambda_2^D(b_{t_0}) < \cdots \) which tends \( +\infty \), and the corresponding eigenfunctions \( \psi_1, \psi_2, \cdots \) are an orthonormal base of \( L^2(0, T) \). Hence, given \( c_i \in R \) and \( y \in H^1_0(0, T) \), we can write \( y(t) = \sum_{i=1}^\infty c_i \psi_i(t) \), and

\[
\int_0^T ((y'(t))^2 - b(t + t_0)y^2(t)) dt = \sum_{i=1}^\infty c_i^2 \int_0^T ((\psi_i'(t))^2 - b(t + t_0)\psi_i^2(t)) dt = \sum_{i=1}^\infty c_i^2 \lambda_1^D(b_{t_0}) \int_0^T \psi_i^2(t) dt.
\]

This completes the proof.

**Lemma 3.3.** There exist \( B_1 > B_2 > 0 \), \( B_3 > 0 \) such that any positive \( T \)-periodic solution \( x(t) \) of (1.1–2.2) satisfies

\[
\|x\|_b < B_2, \quad \|x'\|_b < B_3.
\]

**Proof.** From condition (H_4) and (3.5), we know that there is \( h_0 > 0 \) such that

\[
g(t, s) \leq (b(t) + \varepsilon_0)s + h_0
\]

for all \( t \) and \( s > 0 \).

Multiplying (3.3) by \( x(t) \) and then integrating over \([0, T]\), we get

\[
\|x\|^2_2 = \tau \int_0^T -(xx'' + Cxx') dt = \tau \int_0^T g(t, x(t))x(t) dt \leq \int_0^T \int_0^T ((b(t) + \varepsilon_0) x'(t) + h_0)x(t) dt + \varepsilon_0 \|x\|^2_2 + h_0 \|x\|.
\]

(3.10)

Note from Lemma 3.1 that there exists \( t_0 \) satisfies \( B_0 < x(t_0) < B_1 \). Let \( u(t) = x(t + t_0) \), then \( u \in H^1_0(0, T) \). Thus

\[
\int_0^T b(t)x'^2(t) dt = \int_0^T b(t + t_0)x'^2(t + t_0) dt = \int_0^T b(t + t_0)(x'(t + t_0) + 2x(t_0)u(t) + u''(t)) dt \leq
\]

\[
\int_0^T b(t + t_0)x'(t + t_0) dt + \int_0^T b(t + t_0)(2x(t_0)u(t) + u''(t)) dt \leq \|x\|^2_2 + h_0 \|x\|.
\]
\[
\|u\|_{2}^{2} \leq A_0 + A_1 \|u\|_{2} + \varepsilon_0 \|u\|_{2}^{2} + \int_{0}^{T} b(t + t_0)u^2(t)\,dt,
\]

By the Hölder inequality, the other terms in (3.10) can be estimated as follows:

\[
\varepsilon_0 \|x\|_{2}^{2} \leq \varepsilon_0 \left( TB_1^2 + 2B_1 T^2 \|u\|_{2} + \|u\|_{2}^{2}\right), \quad h_0 \|x\|_{2} \leq h_0 \left( TB_1 + T^2 \|u\|_{2}\right).
\]

Thus (3.10) reads as

\[
\|u\|_{2}^{2} \leq A_0 + A_1 \|u\|_{2} + \varepsilon_0 \|u\|_{2}^{2} + \int_{0}^{T} b(t + t_0)u^2(t)\,dt,
\]

where \( A_0 = \varepsilon_0 TB_1^2 + h_0 TB_1 + B_1^2 \|B\|_{2}, \quad A_1 = 2\varepsilon_0 B_1 T^2 + h_0 T^2 + 2B_1 \|B\|_{2} \) are positive constants.

Now, using \( \int_{0}^{T} b(t + t_0)u^2(t)\,dt \leq \int_{0}^{T} (u^2(t) - b(t + t_0)u^2(t))\,dt \), we get from (3.11) that

\[
\|x\|_{2}^{2} \leq A_1 \|x\|_{2} + A_0.
\]

Consequently, \( \|u\|_{2} < A_2 \) for some \( A_2 > 0 \). By (3.11), one has \( \|x\|_{2} = \|u\|_{2} < A_2 \) for some \( A_2 > 0 \). From these, for any \( t \in [t_0, t_0 + T] \), \( x(t) = \left| x(t_0) + \int_{0}^{T} x(t)\,dt \right| \leq B_1 + T^2 \|x\|_{2} \leq B_1 + T^2 A_2 = B_2 \). Thus \( \|x\|_{\infty} < B_2 \) is obtained.

In order to prove \( \|x\|_{\infty} < B_3 \), we write (1.1) as \( -x'(t) = H(t) := Cx(t) + g(t, x(t)) \). As \( \int_{0}^{T} H(t)\,dt = 0 \), thus \( \|H(t)\|_{2} = 2\|H'(t)\|_{2} \). From (3.9) we have

\[
H'(t) = \max(H(t), 0) \leq C x(t) + \varepsilon_0 x(t) + h_0 \leq C x(t) + C_1,
\]

where \( \|x\|_{\infty} < B_2 \) is used. As \( x(t_1) = 0 \) for some \( t_1 \), we have

\[
\|x\|_{\infty} = \max_{0 \leq s \leq T} \left| x'(s) \right| = \max_{0 \leq s \leq T} \left| \int_{0}^{s} x'(s)\,ds \right| \leq \int_{0}^{T} |H(s)|\,ds = 2\int_{0}^{T} |H'(s)|\,ds \leq 2 \left( C \|x\|_{2} + T C_1 \right) \leq 2 \left( C A_1 T^2 + T C_1 \right) := B_3.
\]

We have proved that the \( W^{2,1} \) norms of \( x \) are bounded.

Next, the positive lower estimates \( m = \min_{t \in [0, T]} x(t) \) are obtained from \( (H_1) \).

**Lemma 3.4.** There exists a constant \( B_4 \in (0, B_0) \) such that any positive solution \( x(t) \) of (1.1–2.2) satisfies

\[
x(t) > B_4 \text{ for all } t.
\]

**Proof.** From \( (H_1) \), we fix some \( A_4 \in (0, B_0) \) such that \( g(t, s) < -CB_4 \) for all \( t \) and all \( 0 < s \leq A_4 \), where \( B_4 \) is same as above. Assume now that \( m = \min_{t \in [0, T]} x(t) = x(t_2) < A_4 \). By Lemma 3.1, \( \max_{t \in [0, T]} x(t) > B_0 \). Let \( t_3 > t_2 \) be the first time instant such that \( x(t) = A_4 \). Then for any \( t \in [t_2, t_3] \), we have \( x(t) \leq A_4 \) and \( -C x'(t) \leq CB_4 \). Hence, for \( t \in [t_2, t_3] \), \( x'(t) = Cx(t) - g(t, x(t)) > CB_4 - C x(t) \geq 0 \). As \( x(t_2) = 0, x'(t) > 0 \) for \( t \in [t_2, t_3] \). Therefore, the function \( x : [t_2, t_3] \to R \) has an inverse, denoted by \( \xi \).

Now multiplying \( (1.1) \) by \( x'(t) \) and integrating over \([t_2, t_3] \), we get

\[
\int_{t_2}^{t_3} -g(\xi(x), x)\,dx = \int_{t_2}^{t_3} -g(t, x(t))x'(t)\,dt = \int_{t_2}^{t_3} (x'(t) + Cx(t) x'(t))\,dt =
\]

\[ \frac{1}{2} \int_{t_1}^{t_2} (x'(t))^2 \, dt + \frac{1}{2} (x'(t))^2 \leq A_4 \]

for some \( A_4 > 0 \). Using condition \((H_1)\)

\[ \int_m^a g(x(t), x) \, dx \geq \int_m^a g_0(x) \, dx \to +\infty \]

(3.12)

if \( m \to 0^+ \). Thus we know from (3.12) that \( m > B_1 \) for some constant \( B_1 > 0 \).

Now we give the proof of Theorem 1.1. From (3.5), there exist constant \( B_i > 0 \) such that

\[ \frac{1}{T} \int_0^T g(t, x) \, dt > 0 \]

for all \( x > B_i \). Let the open bounded in \( X \) be

\[ \Omega = \{ x \in X : E_1 < x(t) < E_2 \text{ and } |x'(t)| < B_2 \text{ for all } t \in [0, T] \}, \]

where \( 0 < E_1 < \min \{ B_0, B_2 \}, E_2 > \max \{ B_1, B_2 \} \). From (3.1), (3.2), it follows that \( QN \) and \( K(I-Q)N \) are continuous, and \( QN(\Omega) \) is bounded and then \( K(I-Q)N(\Omega) \) is compact for any open bounded \( \Omega \subset X \) which means \( N \) is L-compact on \( \Omega \), condition (i) of Lemma 2.2 is satisfied. For any \( x \in \text{Ker}L \cap \partial \Omega \), we have

\[ QN = \frac{1}{T} \int_0^T g(t, x) \, dt \]

and we obtain \( QN(E_1) = \frac{1}{T} \int_0^T g(t, E_1) \, dt < 0, QN(E_2) = \frac{1}{T} \int_0^T g(t, E_2) \, dt > 0 \), which implies the condition (ii) of Lemma 2.2 is satisfied. Define

\[ H(x, \mu) = \mu x + (1-\mu)QN x = \mu x + (1-\mu) \frac{1}{T} \int_0^T g(t, x) \, dt. \]

\( H(x, \mu) \) is a homotopy mapping for all \( (x, \mu) \in (\partial \Omega \cap \text{Ker}L) \times [0,1] \) and by using homotopy invariance theorem (we refer to [11, Section 2.3] for the notion of homotopy mapping and homotopy invariance theorem), we have

\[ D_q(QN |_{\text{Ker}L}, \Omega \cap \text{Ker}L) = D_q(x, \Omega \cap \text{Ker}L) \neq 0. \]

Thus condition (iii) of Lemma 2.2 is also verified. Therefore \( Lx + Nx = 0 \) has at least one solution in \( \Omega \), which means (1.1) has at least one positive T-periodic solution.

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