THE PUBLISHING HOUSE
OF THE ROMANIAN ACADEMY

PROCEEDINGS OF THE ROMANIAN ACADEMY, Series A,
Volume 18, Number 4/2017, pp. 331–336

OPTICAL SOLITON PERTURBATION WITH QUADRATIC-CUBIC NONLINEARITY BY SEMI-INVERSE VARIATIONAL PRINCIPLE

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Abstract. We obtain exact optical soliton solutions to perturbed nonlinear Schrödinger's equation with quadratic-cubic nonlinearity by the aid of semi-inverse variational principle. The perturbation terms include inter-modal dispersion, higher-order dispersions, nonlinear dispersion and self-steepening term, the last two being with full nonlinearity.

Key words: bright soliton, semi-inverse variational principle, quadratic-cubic law.

1. INTRODUCTION

Optical solitons is a treasure trove in the area of fiber optic communications technology [1–23]. Several results in this field are visible across a wide variety of journals. The governing model that describes the propagation of optical solitons and optical soliton complexes (soliton "molecules") is the nonlinear Schrödinger's equation (NLSE) that comes with various forms of nonlinearity. A new form of nonlinearity was proposed during 2011, which is called quadratic-cubic (QC) [5]. Thus far, NLSE with QC nonlinearity has been studied, without prerturbation terms, by the aid of some integration algorithms, including the application of semi-inverse variational principle (SVP) [3, 11]. The parameter dynamics was also retrieved with variational principle [5]. This paper will address the perturbed NLSE with QC form of nonlinearity by the application of SVP. This will reveal bright soliton solutions. The perturbation terms that will be studied are the inter-modal dispersion, third- and fourth-order dispersions (3OD and 4OD) as well as self-steepening term along with nonlinear dispersion. The last two perturbation terms are treated with full nonlinearity.

It must be noted that a similar model has been studied in the past. The results for NLSE with QC nonlinearity, by the aid of SVP, without any perturbation terms have been reported during 2017 [3]. However, in presence of these perturbation terms, the results for NLSE with parabolic and dual-power laws of nonlinearity have been reported using SVP during 2014 [2]. The novelty of this paper lies in the fact that NLSE is studied with QC nonlinearity, in presence of perturbation terms, using SVP. Thus, this work is a generalization and extension of the previously established results [2, 3].

2. GOVERNING MODEL

The governing resonant NLSE with perturbation terms that is studied in nonlinear optics is given in its dimensionless form as [2, 3, 15]:

$$\mathrm{i}q_t + aq_{xx} + \left(b_1|q| + b_2|q|^2\right)q = \mathrm{i}\left[\alpha q_x - \gamma q_{xxx} - \mathrm{i}\sigma q_{xxxx} + \lambda\left(|q|^{2m}q\right)_x + \theta\left(|q|^{2m}\right)_x q\right]. \tag{1}$$

In Eq. (1), the independent variables x and t represent spatial and temporal variables, respectively. The dependent variable $q(x,t)$ gives the complex-valued wave profile and $\mathrm{i}=\sqrt{-1}$. The coefficient of the real-valued constant a is group velocity dispersion (GVD). The nonlinear terms are given by the coefficients of b_1 and b_2, which represent quadratic and cubic nonlinearities, respectively. On the right hand side α is the coefficient of inter-modal dispersion. It occurs when the group velocity of light propagating in multimode optical fibers (or other optical waveguides) depends on the optical frequency and the propagation mode involved. The coefficients of γ and σ are 3OD and 4OD, respectively. These appear when GVD is low and thus the higher-order dispersions compensate for it to maintain the necessary balance between dispersion and nonlinearity for the formation of optical solitons. The coefficient of λ is due to self-steepening that is included to avoid the formation of shock waves. Finally, θ is the nonlinear dispersion. The index $m>0$ is the full nonlinearity parameter.

3. SEMI-INVERSE VARIATIONAL PRINCIPLE

To solve Eq. (1) by SVP, the starting assumption is [2, 3]:

$$q(x,t)=g(s)\mathrm{e}^{\mathrm{i}\varphi(x,t)} \tag{2}$$

where

$$s=x-vt \tag{3}$$

and the phase ϕ is:

$$\phi(x,t)=-\kappa x+\omega t+\theta_0. \tag{4}$$

In Eqs. (2) and (3), $g(x,t)$ represents the amplitude component of the wave and v is the speed of the wave. In Eq. (4), κ represents the soliton frequency, ω is the wave number, and θ_0 is the phase constant.

Substituting (2) into (1) and equating real and imaginary parts leads to [2, 15]

$$\sigma g^{(iv)}-P_2 g''+P_1 g-\left(b_1 g+b_2 g^2\right)g+\lambda\kappa g^{2m+1}=0 \tag{5}$$

and

$$\left(v+2a\kappa+\alpha+3\gamma\kappa^2+4\sigma\kappa^3\right)g'-\left(\gamma+4\sigma\kappa\right)g''+\left\{(2m+1)\lambda+2m\theta\right\}g^{2m}g'=0. \tag{6}$$

Here the notations $g'=\mathrm{d}g/\mathrm{d}s$ and $g''=\mathrm{d}^2g/\mathrm{d}s^2$ etc. are adopted. Here, in Eq. (5)

$$P_1=\omega+a\kappa^2+\alpha\kappa+\gamma\kappa^3+\sigma\kappa^4, \tag{7}$$

$$P_2=a+3\gamma\kappa+6\sigma\kappa^3. \tag{8}$$

From Eq. (6), setting the coefficients of linearly independent functions to zero gives:

$$v=-2a\kappa-\alpha-3\gamma\kappa^2-4\sigma\kappa^3, \tag{9}$$

$$\gamma+4\sigma\kappa=0, \tag{10}$$

and

$$(2m+1)\lambda+2m\theta=0. \tag{11}$$

Thus, equation (9) gives the speed of the soliton in presence of the perturbation terms while relations (10) and (11) are the constraints on the perturbation parameters.

Next, multiplying both sides of (5) by g' and integrating yields

$$\sigma\left(g''\right)^2-2\sigma g'g'''+P_1 g^2-P_2\left(g'\right)^2-2\left(\frac{b_1 g^3}{3}+\frac{b_2 g^4}{4}\right)+\frac{\lambda g^{2m+2}}{m+1}=K, \tag{12}$$

where K is the integration constant. The stationary integral is then defined as [2–4, 6, 10, 11]:

$$J=\int_{-\infty}^{\infty}K\,\mathrm{d}s=\int_{-\infty}^{\infty}\left[3\sigma\left(g''\right)^2+P_1 g^2-P_2\left(g'\right)^2-\frac{2b_1 g^3}{3}-\frac{b_2 g^4}{2}+\frac{\lambda\kappa g^{2m+2}}{m+1}\right]\mathrm{d}s. \tag{13}$$

Now choose [3, 5, 15]

$$g(s)=\frac{A}{D+\cosh(Bs)}, \tag{14}$$

where A is the soliton amplitude, B is its width and D is an external parameter. SVP states that the solution of the perturbed equation (1) will have the same structure as its homogeneous counterpart. But, its amplitude and width will vary according to the coupled system of equations [1–4, 6, 10, 11]:

$$\frac{\partial J}{\partial A}=0, \tag{15}$$

and

$$\frac{\partial J}{\partial B}=0. \tag{16}$$

Substituting (14) into (13) and performing the integrations gives

$$J=\sigma N A^2 B^3+\frac{P_1 A^2 M_6}{3B}-\frac{P_2 A^2 B M_7}{15}-\frac{4b_1 A^3 M_8}{45B}-\frac{b_2 A^4 M_3}{35B}+\frac{2\lambda\kappa A^{2m} M_9\, m(2m+1)}{2^{2m} B(m+1)(4m+1)(4m+3)}\frac{\Gamma(2m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2m+\frac{1}{2}\right)}, \tag{17}$$

where

$$N=\frac{12M_1+7M_2+6M_3-12M_4-8M_5}{35} \tag{18}$$

and the rest of $M_j\left(1\le j\le 9\right)$ are given in Table 1

Table 1

Definition of $M_j\left(1\le j\le 9\right)$	
M_1	$F\left(6,\,2;\,\frac{9}{2};\frac{1-D}{2}\right)$
M_2	$F\left(4,\,2;\,\frac{7}{2};\,\frac{1-D}{2}\right)$
M_3	$F\left(4,\,4;\,\frac{9}{2};\,\frac{1-D}{2}\right)$
M_4	$F\left(5,\,2;\,\frac{9}{2};\frac{1-D}{2}\right)$
M_5	$F\left(5,\,3;\,\frac{11}{2};\,\frac{1-D}{2}\right)$
M_6	$F\left(2,\,2;\,\frac{5}{2};\frac{1-D}{2}\right)$
M_7	$F\left(4,\,2;\,\frac{7}{2};\,\frac{1-D}{2}\right)$
M_8	$F\left(3,\,3;\,\frac{7}{2};\,\frac{1-D}{2}\right)$
M_9	$F\left(2m+2,2m+2;\,2m+\frac{5}{2};\frac{1-D}{2}\right)$

Here, the Gauss' hypergeometric function is defined as

$$F\left(c_1, c_2; c_3; z\right) = \sum_{n=0}^{\infty} \frac{(c_1)_n (c_2)_n}{(c_3)_n} \frac{z^n}{n!}, \tag{19}$$

with the Pochhammer symbol given by

$$\left(p\right)_n = \begin{cases} 1 & n = 0, \\ p(p+1)\cdots(p+n-1) & n > 0. \end{cases} \tag{20}$$

The convergence criterium for hypergeometric function is

$$|z| < 1. \tag{21}$$

which, from the table amounts to saying

$$-1 < D < 3. \tag{22}$$

Furthermore, Rabbe's criteria of convergence implies

$$c_3 < c_1 + c_2 \tag{23}$$

and this is satisfied for all of the hypergeometric functions listed in the table.

Next, for J given by (17), equations (15) and (16) reduce to

$$\sigma N B^4 + \frac{P_1 M_6}{3} - \frac{P_2 B^2 M_7}{15} - \frac{2 b_1 A M_8}{15} - \frac{2 b_2 A^2 M_3}{35} + \frac{2\lambda\kappa A^{2m} M_9 m(2m+1)}{2^{2m}(4m+1)(4m+3)} \frac{\Gamma(2m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2m+\frac{1}{2}\right)} = 0, \tag{24}$$

and

$$3\sigma N B^4 - \frac{P_1 M_6}{3} - \frac{P_2 B^2 M_7}{15} + \frac{4 b_1 A M_8}{45} + \frac{b_2 A^2 M_3}{35} - \frac{2\lambda\kappa A^{2m} M_9 m(2m+1)}{2^{2m}(m+1)(4m+1)(4m+3)} \frac{\Gamma(2m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2m+\frac{1}{2}\right)} = 0, \tag{25}$$

where, $\Gamma(\mathrm{x})$ is Euler's gamma function. Upon uncoupling, (24) and (25) lead to a biquadratic equation for the width B in terms of the soliton amplitude A, which is given by

$$4\sigma N B^4 - \frac{2 P_2 B^2 M_7}{15} - \frac{2 b_1 A M_8}{45} - \frac{b_2 A^2 M_3}{35} + \frac{2\lambda\kappa A^{2m} M_9 m^2 (2m+1)}{2^{2m}(4m+1)(4m+3)} \frac{\Gamma(2m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2m+\frac{1}{2}\right)} = 0, \tag{26}$$

Equation (26) solves to

$$B = \sqrt{\frac{1}{8\sigma N}\left[\frac{2P_2M_7}{15} + \left\{\frac{4P_2^2M_7^2}{225} + 16\sigma N\left(\frac{2b_1AM_8}{45} + \frac{b_2A^2M_3}{35} - \frac{2\lambda\kappa A^{2m}M_9m^2(2m+1)}{2^{2m}(4m+1)(4m+3)}\frac{\Gamma(2m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2m+\frac{1}{2}\right)}\right)\right\}^{\frac{1}{2}}\right]}. \tag{27}$$

This relation between the soliton amplitude and width will exist provided

$$\sigma N\left[\frac{2P_2M_7}{15} + \left\{\frac{4P_2^2M_7^2}{225} + 16\sigma N\left(\frac{2b_1AM_8}{45} + \frac{b_2A^2M_3}{35} - \frac{2\lambda\kappa A^{2m}M_9m^2(2m+1)}{2^{2m}(4m+1)(4m+3)}\frac{\Gamma(2m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2m+\frac{1}{2}\right)}\right)\right\}^{\frac{1}{2}}\right] > 0, \tag{28}$$

and

$$P_2^2 M_7^2 + 900\sigma N\left\{\frac{2 b_1 A M_8}{45} + \frac{b_2 A^2 M_3}{35} - \frac{2\lambda\kappa A^{2m} M_9 m^2 (2m+1)}{2^{2m}(4m+1)(4m+3)} \frac{\Gamma(2m)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(2m+\frac{1}{2}\right)}\right\} > 0. \tag{29}$$

Thus, finally 1-soliton solution to the perturbed NLSE (1), with QC nonlinearity and fully nonlinear perturbation terms, is

$$q(x,t)=\frac{A}{D+\cosh[B(x-vt)]}\mathrm{e}^{\mathrm{i}\left(-\kappa x+\omega t+\theta_0\right)}, \tag{30}$$

where the amplitude-width relation as well as the soliton speed are all indicated above along with the parameter restrictions for the existence of bright soliton.

Figure 1 shows the profile of bright 1-soliton solution to the model. The parameter values chosen in this case are: $a=1/2$, $b_1=b_2=1$, $\alpha=\gamma=1/2$, $\sigma=-1$, $\lambda=\theta=1$, $D=2$, and $m=1$.

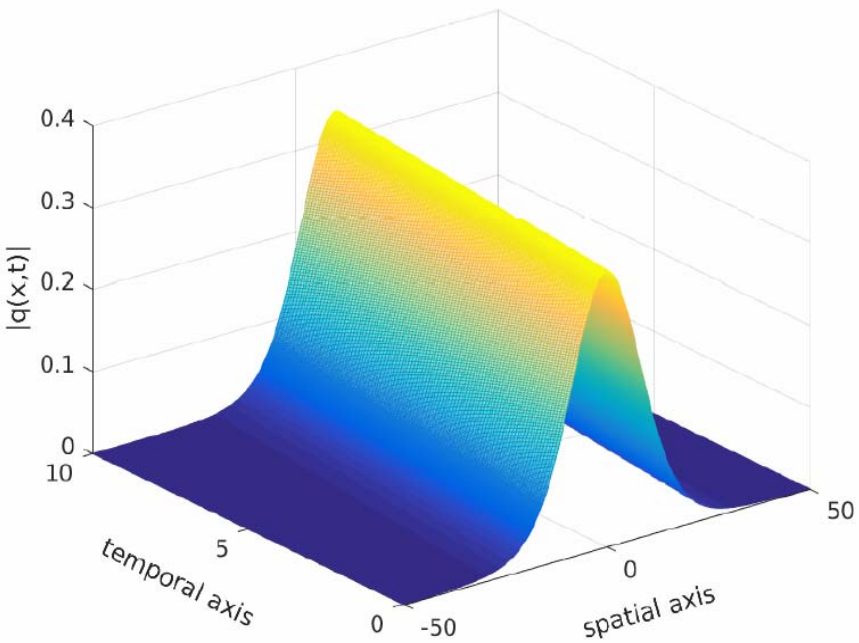

Fig. 1

4. CONCLUSION

This paper retrieved bright optical soliton solutions to the perturbed NLSE that is studied with QC nonlinearity. A numerical simulation of the bright single-soliton solution is also displayed. The results are in terms of Gauss' hypergeometric functions. The perturbed NLSE is otherwise not integrable with the aid of any other known methods that have been developed thus far in the literature. Although explicit expressions for the soliton amplitude and width are not available for this model with SVP, it is the amplitude-width relation one has to stay contended with. The results of this paper carry a lot of scope in future. This integration scheme can be applied to other situations such as optical metamaterials, optical switching, and DWDM systems. These applications will be studied and their results will be available down the road.

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Received May 8, 2017