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REPRESENTATION OF A PREFERENCE RELATION ON CONVEX METRIC SPACES BY A NUMERICAL FUNCTION

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Abstract. In this work, we study the interplay of ordered relation and the structure of the convex metric space on which it is defined: First, we show when a convex metric space is isomorphic with a convex subset of some vector space. Second, we obtain conditions for a representation of a preference relation on a convex metric space.

***Key words**:* representation, order, convex, metric space.
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1. INTRODUCTION

An order is a mathematical formalization of the intuitive notion of the preferences, some of whose members may be interconnected. The representation theory study the existence and construction of real valued order preserving mappings on an ordered universe. Cantor [10] made early achievement in this direction. Next important milestone on the road was set by Debreu (see [12, 13]) who added continuity to order preservation. Later on significant contributions in the field of existence and construction of real valued order preserving functions were made by Beg and Samina [6], Bosi and Isler [7] and Campion et al. [9] These achievements made representation theory a significant area of Mathematics having applications in Multi Criteria Decision Problems, Economics, Game Theory and Social Choice Theory.

Takahashi [18] introduced the notion of convex metric spaces by generalizing the concept of convex functions to metric spaces. He studied properties of convex metric spaces and gave several examples including examples of convex metric space which is not isomorphic with any convex subset of a normed space. Shimizu and Takahashi [17] gave the idea of uniformly convexity in convex metric spaces and obtained its properties and constructed example of a uniformly convex metric space which is not a normed space. Beg [2, 3] established an inequalities in uniformly convex complete metric spaces analogous to the parallelogram law in Hilbert spaces and further exploited it to obtain that every compact convex subset of a uniformly convex complete metric space is Chebyshev. Afterwards Abdelhakim [1], Berinde and Pacurar [8], Gabeleh and Shahzad [14], Ghanifard et al. [15] and Kumar and Tas [16] used these notions to obtain fixed point under different conditions. Recently Beg [5, 6] presented the idea of ordered convex metric spaces and ordered uniform convexity in ordered convex metric spaces. In all these papers the major focus was on fixed point. The aim of present work is to study the interplay of ordered relation and the structure of the convex metric space on which it is defined: First, we show when a convex metric space is isomorphic with a convex subset of some vector space. Second, we obtain conditions for a representation of a preference relation on a convex metric space.

This paper is organized as follows: Introduction of paper is given in Section 1. Section 2 presents some basic definitions and notions about convex metric spaces and ordered binary relations. In Section 3, we proved that a convex metric space (X,d,W) having property $(\mathscr{L})$ is isomorphic to a convex subset of some real vector space. Also, in this section we obtain conditions for the representation of a preference relation $\preceq$ on a convex metric space (X,d,W) by a real valued function. Conclusion is given in Section 4.

2. PRELIMINARIES

In this section basic notations and fundamental notions related to convex structure on metric spaces and binary order relation are reviewed and presented for subsequent use.

Definition 1 ([18]). Let (X,d) be a metric space and $I=[0,1]$. A mapping $W: X\times X\times I\to X$ is said to be a convex structure on X if for each $(x,y,\alpha)\in X\times X\times I$ and $u\in X$,

$$d(u,W(x,y,\alpha))\leq \alpha d(u,x)+(1-\alpha)d(u,y). \tag{1}$$

A metric space X together with a convex structure W is said to be a convex metric space. A subset $M\neq\phi$ of X is called convex if $W(x,y,\alpha)\in M$ for all $(x,y,\alpha)\in M\times M\times I$.

Remark 1 ([1,3,18]). The convex metric space X has the properties;.i. $W(x,y,1)=x$, ii. $W(x,y,0)=y$, iii. $W(x,x,\alpha)=x$.

For more examples of convex metric spaces in the existing literature (see [1,2,17,18]).

Definition 2 ([4]). A convex metric space X is said to have property $(\mathscr{L})$ if for all $x,y,z\in X$ and α,β,γ in I, we have
i. $W(W(x,y,\alpha),W(x,y,\beta),\gamma)=W(x,y,\alpha\gamma+(1-\gamma)\beta$
ii. $W(x,y,\alpha)=W(y,x,1-\alpha)$.

Remark 2. Taking $\beta=0$ in Definition 2(i), we obtain

$$W(W(x,y,\alpha),y,\gamma)=W(x,y,\alpha\gamma). \tag{2}$$

Each normed space has property $(\mathscr{L})$, if we define $W(x,y,t)=tx+(1-t)y$.

Definition 3 ([11]). Let $\preceq$ be a binary relation on an arbitrary nonempty set X and $x,y,z\in X$. The binary relation $\preceq$ is called ordered relation if it satisfies; (i). $x\preceq x$, (ii). $x\preceq y$ and $y\preceq z$ implies $x\preceq z$, (iii) $x\preceq y$ and $y\preceq x$ implies $x=y$. A reflexive and transitive relation $\preceq$ is called pre-order.

Remark 3 ([11]). Let $\preceq$ be an ordered relation on a set X. By $a\prec b$ we mean $a\preceq b$ and $a\neq b$. The inverse of $\preceq$ is defined as $a\succcurlyeq b$ if $b\preceq a$.Transitivity of order relation $\preceq$ implies $a\not\prec b\not\prec c\Longrightarrow a\not\prec c$ for all $a,b,c\in X$.

Definition 4 ([11]). An ordered set is called totally ordered if it has no incomparable elements.

LEMMA 1 ([19]). ***Zorn's lemma***. *If X is a non empty ordered set such that every totally ordered subset of X has an upper bound in X, then X contains a maximal element.*

Definition 5 ([4]). An ordered relation $\preceq$ on a convex metric space X is said to be continuous if the sets $\{\alpha: W(x,y,\alpha)\preceq z\}$ and $\{\alpha: W(x,y,\alpha)\succeq z\}$ are closed for $x,y,z\in X$.

Definition 6 ([4]). A metric space (X,d) with a convex structure W and continuous order relation $\preceq$ is said to be an ordered convex metric space, and we denote it by $(X,d,W,\preceq)$.

3. REPRESENTATION

Definition 7. A function h from a convex metric space (X,d,W) to a real vector space is said to be convexity preserving (CP) if $h(W(x,y,\alpha)) = \alpha h(x) + (1-\alpha)h(y)$.

The set of all real valued CP functions on (X,d,W) is denoted by $CP(X)$. Obviously, $CP(X)$ is a vector space. We can define an equivalence relation $\sim$ on a convex metric space (X,d,W) by;

$$x \sim y \quad \Leftrightarrow \quad \forall h \in CP(X), \quad h(x) = h(y).$$

Then this equivalence relation $\sim$ is compatible with convex metric space structure and induced convexity operation on equivalence classes $\widetilde{x}, \widetilde{y}$, $W(\widetilde{x},\widetilde{y},\alpha) = W(x,y,\widehat{\alpha})$ for which $X/\sim$ is itself a convex metric space.

Definition 8. A function $F : (X,d,W,\preceq) \to \mathbb{R}$ represents the order $\preceq$ if for all x,y in X,

$$x \preceq y \Leftrightarrow F(x) \leq F(y). \tag{3}$$

Function F is called a representation of order (preference relation) $\preceq$.

THEOREM 1. *Let (X,d,W) be a convex metric space having property $(\mathscr{L})$.If*

$$W(W(x,y,\alpha),z,\gamma) = W(x,W(y,z,\frac{\gamma(1-\alpha)}{1-\alpha\gamma}),\alpha\gamma) \;\; for \;\; \alpha\gamma \neq 1, \tag{4}$$

and

$$W(x,y,\alpha) = W(x,z,\alpha) \Rightarrow y = z \tag{5}$$

then X is isomorphic to some convex subset of a real vector space.

Proof. In case X is singleton proof is obvious. So assume that X has more then one element. Take $u \neq v$ in X; now we construct a function $g \in CP(X)$ such that $g(u) \neq g(v)$. For any $g(u) \neq g(v)$, define the subset

$$X_{u,v} = \{z \in X : \exists \alpha \in [0,1] \text{ and } z = W(u,v,\alpha)\}.$$

Now using Definition 2 and Remark 2, we obtain

$$\forall z,z^{'} \in X_{u,v}, \quad \forall \gamma \in [0,1], \quad W(z,z^{'},\gamma) = W(u,v,\alpha\gamma + (1-\gamma)\beta),$$

so that $X_{u,v}$ is a convex metric space. Next for any $z = W(u,v,\alpha)$, define

$$g(z) = \alpha g(u) + (1-\alpha)g(v).$$

Remark 1 and Equality 5 implies that g is a well defined CP function on $X_{u,v}$.

Next we want to extend g from $X_{u,v}$ to a convex superset X_x $(i.e.\ X_{u,v} \subset X_x \subset X)$, in such a way that g is CP function on X_x. Let $x \in X \backslash X_{u,v}$, and we show that,

$$X_x = \{z \in X : \exists \alpha \in [0,1], \quad \exists y \in X_{u,v} \quad \text{and } z = W(x,y,\alpha)\}$$

is a convex subset and that g can be extended to a CP function on X_x. Choose any two elements $z = W(x,y,\alpha)$ and $z^{'} = W(x,y^{'},\beta)$ in X_x. If $\gamma,\alpha,\beta \notin \{0,1\}$ then using Definition 2 we have

$$\begin{aligned} W(z,z^{'},\gamma) &= W(W(x,y,\alpha),W(x,y^{'},\beta),\gamma) \\ &= W(x,W(y,y^{'},\frac{\gamma(1-\alpha)}{1-\alpha\gamma-\beta(1-\gamma)}),\beta(1-\gamma)+\alpha\gamma). \end{aligned}$$

Therefore $W(z,z',\gamma) \in X_x$. It also hold when $\gamma,\alpha,\beta \in \{0,1\}$.Thus X_x is a convex set. Next, fix some value for $g(x)$, and for any $z = W(x,y,\alpha)$, define

$$g(z) = \alpha g(x) + (1-\alpha)g(y).$$

To check that g is a well defined function, first note that if $z = W(x,y,\alpha) = W(x,y',\beta)$,then Equality 5 implies that either $y = y'$ and $\alpha = \beta$ or $y \neq y'$ and $\alpha \neq \beta$. Consider the case $y \neq y'$ $\alpha \neq \beta$.

Assume that $\alpha < \beta$. Then property $(\mathscr{L})$ implies that

$$y = W(x,y',\frac{\beta-\alpha}{1-\alpha}).$$

It further implies that $g(z)$ has unique value. Obviously g is also CP on X_x.

Let $C = \{M : M$ is any convex subset of X containing x,y and the function g has been extended to a CP function g_m on $M\}$. Now we can introduce the partial order on the set $\mathbb{C} = \{(M,g_m) : M \in C\}$ by

$$(M,g_m) \preceq (M',g_{m'}) \quad \Leftrightarrow \quad M \subset M' \text{ and } g_{m'} \text{ extends } g_m.$$

Now take any totally ordered chain $(M_i,g_{m_i})_{i\in I}$ of $\mathbb{C}$. This chain $(M_i,g_{m_i})_{i\in I}$ has an upper bound $(\bigcup_{i\in I} M_i,h)$ with h defined by $h(x) = g_{m_i}(x)$ if x is in M_i. Zorn's lemma 1 further implies that $\mathbb{C}$ has a maximal point (say) $(\mathbf{M},h)$ for $\preceq$. Suppose $\mathbf{M} \neq X$.Take $x \notin \mathbf{M}$ and construct $(\mathbf{R},v)$ such that $(\mathbf{M},h) \subset (\mathbf{R},v)$ and $\mathbf{R} \neq \mathbf{M}$ a contradiction. Hence $\mathbf{M} = X$ and h extends g to X. □

LEMMA 2. *Let $(X,d,W,\preceq)$ be a totally ordered convex metric space having property $(\mathscr{L})$. Then X is isomorphic to a convex subset of some vector space Z.*

Proof. Assume that X is a convex subset of some vector space Z, and $W(x,y,\alpha) = \alpha x + (1-\alpha)y$ for all x,y in X and $\alpha \in I$.Choose u,v in X such that $u \prec v$. Now for any z in X one and only one of the following is possible;

a) there exists $\alpha \in (0,1)$ such that $u = W(z,v,\alpha)$,

b) there exists $\alpha \in I$ such that $z = W(u,v,\alpha)$,

c) there exists $\alpha \in (0,1)$ such that $v = W(u,z,\alpha)$.

Since $\preceq$ is total and transitive therefore for any z in X one and only one of the following is possible a$'$) $z \prec u$, b$'$), $u \preceq z \preceq v$ c$'$) $v \prec z$. If a$'$) holds then $z \prec u \prec v$. Now by anti-symmetry and continuity of $\preceq$ and connectedness of I, there exists α in $(0,1)$ such that $W(z,v,\alpha) = u$. In the same way, b$'$) implies that there exists α in I such that $W(u,v,\alpha) = z$ and c$'$) implies that there exists α in $(0,1)$ such that $W(u,z,\alpha) = v$.Thus for any z in X at least one of a), b), and c) holds. To show that only one of these is true, choose z in X. Suppose a) holds and $z \nprec u$. Therefore there exists $\alpha \in (0,1)$ such that $u = W(z,v,\alpha)$ and $z \succeq u$.Thus either $u \prec v \prec z$ or. $u \preceq z \preceq v$. Hence either b$'$) or c$'$) holds. If b$'$) holds then from a) it follows that there exists α in $(0,1)$ such that $u = W(z,v,\alpha)$. Since $u \prec v$ implies $u \neq v$. It further implies that $u \neq z \neq v$. As b$'$) implies b) , thus there exists a $\gamma \in (0,1)$ such that $z = W(u,v,\gamma)$, which further implies $u = W(u,v,\alpha\gamma)$ with $\alpha\gamma \in (0,1)$.Hence a contradiction to $u \neq v$.Similarly c$'$) yields a contradiction. Thus $z \prec u$. Therefore a) and a$'$) are equivalent. In similar manner b) implies $u \preceq z \preceq v$, and c) implies $v \prec z$. Hence, b) and b$'$), c) and c$'$) are equivalent.

To show that Z is one dimensional. Without loss of generality, assume that X is an interval in $\mathbb{R}$ with usual operation $\leq$. As the the operation $\leq$ on $\mathbb{R}$ is total and antisymmetric, thus either $u < v$ or $v < u$. Assume $u < v$. Now choose a pair of distinct points x,y in X, such that $x \prec y$. Then, only one of the following holds; (i) $y \prec u$, (ii) $y = u$, (iii) $u \prec y \prec v$, (iv) $y = v$, (v) $v \prec y$. If (i) is satisfied, then from $y \prec u \prec v$ and a) above it follows that there exists α in $(0,1)$ such that $u = W(y,v,\alpha)$. As $u \prec v$, thus $y \prec u$. Similarly, $x \prec y \prec u$ and a) imply $x < y$. When (ii) is true, then from $x \prec y = u \prec v$ and a) it follow $x < u = y$.If (iii) holds, then it follows from $u \prec y \prec v$ and a) that $y < v$. Thus , $x \prec y \prec v$ and a) imply $x < y$. In case (iv) is satisfied and $x \prec u$, then from $x \prec u \prec v = y$ and a) it follow that $x < y$.If (iv) is satisfied and $u \prec x$, then from $u \prec x \prec v = y$ and a) it follow that $x < y$. When (iv) holds and $x = u$, obviously $x < y$. Next assume (v) holds. Now from $u \prec v \prec y$

and a) it follows that $u < v < y$. In case $x \prec v$, then from $x \prec v \prec y$, a) and $v \prec y$ it follows that $x < y$.In the same manner, if $v \prec x$, then from $v \prec x \prec y$, a) and $v \prec y$ it follows that $x < y$.When $x = v$, then obviously $x < y$. As x, y are chosen arbitrarily, therefore $x' \prec y'$ implies $x' < y'$.

Conversely, if $x \nprec y$ when $x < y$ for some $x, y \in X$.Then completeness and antisymmetry of $\preceq$ imply $y \prec x$ and now above argument implies $y < x$. Thus a contradiction. Hence $<$ and $\prec$ are identical. As $\preceq$ is antisymmetric, thus $\preceq$ and $\leq$ are identical. □

PROPOSITION 1. *Let* $(X, d, W, \preceq)$ *be a totally ordered convex metric space having property* $(\mathscr{L})$*. If* $x \neq y$*, then for all* α *in* $(0,1)$, $x \neq W(x,y,\alpha)$.

Proof. Let $x \neq y$ and $x = W(x,y,\alpha)$ for some α in $(0,1)$. Property $(\mathscr{L})$ and equality 2 imply $x = W(x,y,\alpha) = W(W(x,y,\alpha),y,\alpha) = W(x,y,\alpha^2) = ... = W(x,y,\alpha^n)$ for any $n \in \mathbb{N}$. As $x \neq y$ and $\preceq$ is total therefore either $x \prec y$ or $y \prec x$. In case $y \prec x$, continuity of $\preceq$ implies that $\{\alpha : W(x,y,\alpha) \prec x\}$ is an open set containing zero. Therefore there exists some positive γ such that for any α, $0 \leq \alpha < \gamma$, $W(x,y,\alpha) \prec x$.Therefore for large n we have $\alpha^n < \gamma$. Thus $x = W(x,y,\alpha^n) = y \prec x$. Hence a contradiction. Similarly we can show that the case $x \prec y$ also yields a contradiction. □

PROPOSITION 2. *Let* $(X, d, W, \preceq)$ *be a totally ordered convex metric space having property* $(\mathscr{L})$*. If* $x \neq y$ *and* α, β *in* $[0,1]$*, then* $W(x,y,\alpha) = W(x,y,\beta)$ *imply* $\alpha = \beta$.

Proof. Let x, y in X and α, β in $[0,1]$ be such that $x \neq y$ and $\alpha \neq \beta$ (without loss of generality choose $\beta < \alpha$). Assume $W(x,y,\alpha) = W(x,y,\beta)$. If $\beta = 0$ then from Remark 1, Definition 2 and $x \neq y$ we have $y = W(x,y,\alpha)$ and $\alpha \in (0,1)$. In case $\gamma = 1-\alpha$. Then Definition 2 implies

$$y = W(x,y,\alpha) = W(y,x,1-\alpha) = W(y,x,\gamma),$$

with $\gamma \in (0,1)$. It contradicts Proposition 1. Thus our hypothesis holds when $\beta = 0$. Next assume $\beta > 0$. If $\alpha = 1$ then Remark 1, Definition 2 and $x \neq y$ imply $x = W(x,y,\beta)$ with $\beta \in (0,1)$. It further implies a contradiction with Proposition 1. Finally assume that $0 < \beta < \alpha < 1$.As

$$(1 - \frac{\alpha-\beta}{1-\beta})(1-\beta) = 1-\alpha.$$

Thus by using property $(\mathscr{L})$, we have

$$W(x,y,\alpha) = W(x, W(x,,y,\beta), \frac{\alpha-\beta}{1-\beta}).$$

Choosing $\gamma = \frac{\alpha-\beta}{1-\beta}$ and $z = W(x,y,\beta)$. As $\beta \in (0,1)$. Proposition 1 implies $z \neq x$. Now from $W(x,y,\alpha) = W(x,y,\beta) = z$ it follows that $z = W(z,x,1-\gamma)$. Because $\gamma \in (0,1)$, it contradicts the Proposition 1 . □

PROPOSITION 3. *Let* $(X, d, W, \preceq)$ *be a totally ordered convex metric space having property* $(\mathscr{L})$*. Then for all* x, y, z *in* X *and* α *in* $(0,1)$,

$$W(x,y,\alpha) = W(x,z,\alpha) \Rightarrow y = z. \tag{6}$$

Proof. Choose $x, y, z \in X$ with $y \neq z$. Let $W(x,y,\alpha) = W(x,z,\alpha)$ for some $x \in X$ and $\alpha \in (0,1)$. In case $x = y$, we have $x = W(x,x,\alpha) = W(x,y,\alpha) = W(x,z,\alpha)$. It contradicts Proposition 1. In the same way, $x = z$ implies a contradiction. Therefore $y \neq x \neq z$. Now one and only one of the following is possible;

$$x \prec y \prec z, x \prec z \prec y, z \prec x \prec y, z \prec y \prec x,\ y \prec x \prec z, y \prec z \prec x.$$

When $x \prec y \prec z$, it follows from a) from proof of Lemma 2 that there exist $\gamma \in (0,1)$ such that $y = W(x,z,\gamma)$. Therefore

$$W(x,z,\alpha) = W(x,y,\alpha) = W(x, W(x,z,\gamma), \alpha) = W(x,z,\alpha + (1-\alpha)\gamma).$$

Proposition 2 implies that $\alpha = \alpha + (1-\alpha)\gamma$. As $\alpha \in (0,1)$ it further implies that $\gamma = 0$. Hence a contradiction. Similarly other cases also leads to a contradiction. Hence (6) holds. □

PROPOSITION 4. *Let $(X,d,W,\preceq)$ be a totally ordered convex metric space having property $(\mathscr{L})$. Then for all x,y,z in X and α,γ in $[0,1]$,*

$$W(W(x,y,\alpha),z,\gamma) = W(x,W(y,z,\frac{\gamma(1-\alpha)}{1-\alpha\gamma}),\alpha\gamma) \ \ for \ \ \alpha\gamma \neq 1, \tag{7}$$

Proof. Choose $x,y,z \in X$. In case $x = y$ or $x = z$ or $y = z$, Definition 2(i) implies equality 7. So assume that x,y,z are distinct. Since $\preceq$ is antisymmetric therefore one and only one of following is true:

$$z \prec y \prec x, y \prec z \prec x, z \prec x \prec y, x \prec z \prec y, x \prec y \prec z, y \prec x \prec z.$$

In case $z \prec y \prec x$, it follows from a) from proof of Lemma 2 that there exists $\beta \in (0,1)$ such that $y = W(x,z,\beta)$.Now Definition 2(i) further implies that for $\alpha,\gamma \in [0,1]$ with $\alpha\gamma \neq 1$,

$$\begin{aligned}
W(W(x,y,\alpha),z,\gamma) &= W(W(x,W(x,z,\beta),\alpha),z,\gamma) \\
&= W(W(x,z,\alpha+(1-\alpha)\beta),z,\gamma) \\
&= W(W(x,z,(\alpha+(1-\alpha)\beta)\gamma),W(x,W(y,z,\frac{(1-\alpha)\gamma}{1-\alpha\gamma}),\alpha\gamma) \\
&= W(x,W(W(x,z,\beta),z,\frac{(1-\alpha)\gamma}{1-\alpha\gamma}),\alpha\gamma) \\
&= W(x,W(x,z,\frac{(1-\alpha)\beta\gamma}{1-\alpha\gamma}),\alpha\gamma) \\
&= W(x,z,(\alpha+(1-\alpha)\beta)\gamma).
\end{aligned}$$

Hence equality 7 holds. Other cases can be shown in similar way. □

THEOREM 2. *Let $(X,d,W,\preceq)$ be a totally ordered convex metric space having property $(\mathscr{L})$. Then X is isomorphic to an interval in $\mathbb{R}$. More over binary relation $\preceq$ is equivalent to usual $\leq$ relation.*

Proof. Proposition 4 and Proposition 3 imply that all conditions of Theorem 1 are satisfied. Therefore X is ismorphic to some convex subset of a real vector space. Lemma 1 implies that convex subset is an interval of set of real numbers $\mathbb{R}$ with usual distance d and order $\leq$. □

Example 1. Consider $X = \{(p,r) : p,r$ are real numbers and $p^2 + r^2 \leq 1\}$ with the usual Euclidean distance d, convex structure $W : X \times X \times I \to X$ given by

$$W((p_1,r_1),(p_2,r_2),\alpha) = (\alpha p_1 + (1-\alpha)p_2, \alpha r_1 + (1-\alpha)r_2),$$

and ordered by $(p_1,r_1) \prec (p_2,r_2) \Leftrightarrow p_1 < p_2$ or in case $p_1 = p_2$ then $r_1 < r_2$. Now $(X,d,W,\preceq)$ is a totally ordered convex metric space having property $(\mathscr{L})$ and mapped onto $[-1,1] \subset \mathbb{R}$. More over binary relation $\preceq$ on X is equivalent to usual $\leq$ relation on $\mathbb{R}$.

4. CONCLUSION AND DISCUSSION

We obtained conditions for representation of order relation $\preceq$ defined on a convex metric space. Is it possible to obtain a version of above Theorem 2 for preordered binary relations $\preceq$ (not antisymmetric)? The study of representation of a pre order relation on convex metric space will be of great interest with significant applications. To show applications of these results in multi criteria decision problems, gambling, economics, game theory approximation theory, optimization theory and social choice theory is another challenging task.

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