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A FUNCTIONAL-ANALYTIC CONSTRUCTION OF STOCHASTIC INTEGRALS IN RIEMANNIAN MANIFOLDS

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Abstract. This article presents a construction of the concept of stochastic integration in Riemannian manifolds from a purely functional-analytic point of view. We show that there are infinitely many such integrals, and that any two of them are related by a simple formula. We also find that the Stratonovich and Itô integrals known to probability theorists are two instances of the general concept constructed herein.

Keywords: stochastic integral, Itô integral, Stratonovich integral, Wiener measure, Riemannian manifold.
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1. MOTIVATION AND CONTEXT

The concept of stochastic integral is familiar to most probability theorists, manifesting itself in the guise of its two avatars: the Stratonovich integral and the Itô integral; it is always presented within the conceptual framework of probability theory. The aim of this work is to reconstruct the very same concept solely upon functional-analytic and Riemannian foundations. Not only shall we achieve this goal, but we shall even be able to exhibit an infinite family of such integrals, all of them particular instances of a single underlying general concept; among these we shall also find the two historically important integrals mentioned above. This article is a condensed version, lacking proofs and several notable results, of the significantly more detailed work presented in [11]; in particular, all the proofs of the statements made in this text may be found therein.

In the following, M will be a separable connected Riemannian manifold and $x_0 \in M$ some fixed arbitrary point. If $t > 0$, we shall repeatedly make use of the space $\mathscr{C}_t = \{c : [0,t] \to M \mid c \text{ is continuous, with } c(0) = x_0\}$, that we shall endow with the natural Wiener measure w_t. The form to integrate along curves will be $\alpha \in \Omega^1(M)$, a real smooth 1-form.

In order to connect this article with the stochastic literature, let us briefly recall some elements of stochastic integration in $\mathbb{R}^n$ without any claim of rigour. If c is a smooth enough curve, the Riemann sums

$$\sum_{j=0}^{2^k-1} \alpha\left(c\left(\frac{jt}{2^k}\right)\right)\left[c\left(\frac{(j+1)t}{2^k}\right) - c\left(\frac{jt}{2^k}\right)\right]$$

converge to the line integral $\int_c \alpha$. It is worth asking ourselves: if c is merely continuous (or, even less, only an element of $\prod_{s\in[0,t]} M$), do these sums still converge to something meaningful and useful? The answer is known to be in the affirmative, but in a slightly weaker sense, it no longer being true for every curve: it turns out that for *almost* every curve c (with respect to the Wiener measure), the limit exists and is called the *Itô integral*. Furthermore, if we symmetrize the above Riemann sums, meaning that we should now consider the sums

$$\sum_{j=0}^{2^k-1} \frac{1}{2}\left[\alpha\left(c\left(\frac{jt}{2^k}\right)\right)+\alpha\left(c\left(\frac{(j+1)t}{2^k}\right)\right)\right]\left[c\left(\frac{(j+1)t}{2^k}\right)-c\left(\frac{jt}{2^k}\right)\right],$$

these, too, will converge, for almost every curve c, but this time to a different limit, called the *Stratonovich integral*.

The starting point of our development is the useful remark that, if in the formula

$$\sum_{j=0}^{2^k-1} \int_{[0,1]} \alpha_{(1-\tau)c(\frac{jt}{2^k})+\tau c(\frac{(j+1)t}{2^k})}\left[c\left(\frac{(j+1)t}{2^k}\right)-c\left(\frac{jt}{2^k}\right)\right] \mathrm{d}P(\tau),$$

we take the Borel probability P on $[0,1]$ to be either $P=\delta_0$ (the Dirac probability concentrated in 0) or $P=\frac{1}{2}(\delta_0+\delta_1)$, we obtain precisely the sums seen above that converge to either the Itô or, respectively, the Stratonovich integral. We conclude that these two stochastic integrals and their approximating sums seem to be particular cases of a general, single concept, that we shall indeed construct below. The generalization of this formula from $\mathbb{R}^n$ to M is quite straightforward: the line segment $\tau\mapsto\tau c(\frac{jt}{2^k})+(1-\tau)c(\frac{(j+1)t}{2^k})$ will get replaced by the unique minimizing geodesic between $c(\frac{jt}{2^k})$ and $c(\frac{(j+1)t}{2^k})$ (whenever it exists, of course), and the vector $c(\frac{(j+1)t}{2^k})-c(\frac{jt}{2^k})$ will get replaced by the tangent vector to this geodesic at τ.

Let us consider the trivial vector bundle $M\times\mathbb{C}$, endowed with the usual Hermitian structure, and with the connection $\nabla^{(\alpha)}f=\mathrm{d}f+\mathrm{i}f\alpha$, where $\mathrm{i}=\sqrt{-1}$ is a complex square root of -1. It is easy to see that $\nabla^{(\alpha)}$ is Hermitian, and that the operator $-\Delta^{(\alpha)}=(\nabla^{(\alpha)})^*\nabla^{(\alpha)}:C_0^\infty(M)\to C_0^\infty(M)$ is symmetric and positive-definite. The usual Friedrichs construction will then give us a self-adjoint and positive-definite extension L_α that will be densely defined in $L^2(M)$. Using the results obtained by Batu Güneysu in chapter XI of his monograph [7], the semigroup $(\mathrm{e}^{-tL_\alpha})_{t\geq 0}$ will admit an integral kernel h_α. Using the main theorem in [10] on $(0,\infty)\times M\times M$, the parabolic operator $\partial_t+L_\alpha\oplus L_\alpha$ will be hypoelliptic, whence we deduce that h_α is smooth. The diamagnetic inequality (proposition XI.5 in [7]), then tells us that $|h_\alpha(t,x,y)|\leq h(t,x,y)$ for every $t>0$ and $x,y\in M$, where h is the heat kernel on M.

For every $k\in\mathbb{N}$ we shall consider the natural projection $\pi_k:\mathscr{C}_t\to M^{2^k}$ given by $\pi_k(c)=\left(c(\frac{t}{2^k}),\dots,c(\frac{2^k t}{2^k})\right)$. Regardless of whether we endow $\mathscr{C}_t$ with the topology of uniform convergence of curves, or with the one of pointwise convergence of curves, π_k will be continuous. The continuous functions on some topological space will be denoted by $C(X)$, and the continuous bounded functions by $C_b(X)$. The complex spaces $L^p(X)$ will have the usual meaning for $p\in[1,\infty]$ whenever X is endowed with a measure. The space $L^0(X)$ is the space of complex-valued measurable functions identified under equality almost everywhere; the natural topology upon it is the one of convergence in measure.

In order to ease the reader's navigation through the text that follows, now is the right time to sketch the result that we are looking for, and the strategy that we shall use to obtain it. We shall begin by constructing a very special function $\rho_{\alpha,t}\in L^\infty(\mathscr{C}_t)$, following which we shall show that the map $\mathbb{R}\ni s\mapsto\rho_{s\alpha,t}\in\mathscr{B}(L^2(\mathscr{C}_t))$ (the space of bounded operators in $L^2(\mathscr{C}_t)$) is a strongly continuous 1-parameter unitary group which, by Stone's theorem, will have a self-adjoint generator $\mathrm{Strat}_t(\alpha)$ (which will be later seen to be precisely the Stratonovich stochastic integral, this also justifying its notation). The difficulty in proving this assertion comes from the fact that $\rho_{\alpha,t}$ will be obtained through an abstract procedure which will obscure the group structure and its unitarity. In order to obtain these very concrete properties, we shall construct a sequence of functions that will trivially exhibit them, and which converges to $\rho_{\alpha,t}$; this convergence will transfer these properties to $\rho_{\alpha,t}$.

More precisely, we shall construct a sequence of real measurable functions $S_{P,t,k}(\alpha)$, linear in $\alpha\in\Omega^1(M)$, such that $\mathrm{e}^{\mathrm{i}S_{P,t,k}(\alpha)}\to\rho_{\alpha,t}$ in $L^2(\mathscr{C}_t)$. Although simple, this idea is complicated by technical details that we shall point out when we encounter them, and that force us to approach the problem indirectly: instead of proving the desired convergence directly on $L^2(\mathscr{C}_t)$ (which seems extremely difficult), we shall first prove it in the space $L^2(\mathscr{C}_t(\overline{U}))$ associated to an arbitrary relatively compact open subset U with smooth boundary, following which we shall consider an exhaustion of M with such subsets, which will allow us to prove the convergence in $L^2(\mathscr{C}_t)$.

2. A GENERALIZED WIENER MEASURE

Let $U \subseteq M$ be a connected relatively compact open subset, with (possibly empty) smooth boundary, such that $x_0 \in U$ (if M is compact we shall take $U = M$). We shall endow the space

$$\mathscr{C}_t(\overline{U}) = \{c : [0,t] \to \overline{U} \mid c \text{ is continuous, with } c(0) = x_0\} ,$$

with the corresponding intrinsic Wiener measure $w_t^{(U)}$ (for details about the Wiener measure, the article [1] contains all the necessary constructions and explanations; note that the constructions therein are not probabilistic, but functional-analytic, therefore our project of a purely functional-analytic construction of stochastic integration is not compromised). This is a metric space when endowed with the distance $D(c_0,c_1) = \max_{s\in[0,t]} d\big(c_0(s),c_1(s)\big)$; it is separable (and therefore second-countable) by [9]. In particular, we may use Luzin's theorem on it.

Let

$$\mathrm{Cyl}(\mathscr{C}_t(\overline{U})) = \{f \in C_b(\mathscr{C}_t(\overline{U})) \mid \exists k \in \mathbb{N} \text{ and } f_k \in C(\overline{U}^{2^k}) \text{ such that } f = f_k \circ \pi_k\}$$

be the algebra of continuous cylindrical functions on $\mathscr{C}_t(\overline{U})$. Clearly, $\mathrm{Cyl}(\mathscr{C}_t(\overline{U})) \subset L^1(\mathscr{C}_t(\overline{U}))$.

THEOREM 1. *The algebra* $\mathrm{Cyl}(\mathscr{C}_t(\overline{U}))$ *is dense in* $L^p(\mathscr{C}_t(\overline{U}), w_t^{(U)})$ *for every* $p \in [1,\infty)$.

Let us define the (obviously linear) functional $W_{\alpha,t}^{(U)} : \mathrm{Cyl}(\mathscr{C}_t(\overline{U})) \to \mathbb{C}$ by

$$W_{\alpha,t}^{(U)}(f_k \circ \pi_k) = \int_U \mathrm{d}x_1\, h_\alpha^{(U)}\left(\frac{t}{2^k}, x_0, x_1\right) \cdots \int_U \mathrm{d}x_{2^k}\, h_\alpha^{(U)}\left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k}\right) f_k(x_1,\dots,x_{2^k})$$

for every $f_k \circ \pi_k \in \mathrm{Cyl}(\mathscr{C}_t(\overline{U}))$, where $h_\alpha^{(U)}$ is the integral kernel on U associated to the connection $\mathrm{d} + \mathrm{i}\alpha$ in the trivial bundle $U \times \mathbb{C}$, constructed as explained above (again, for details see chapter XI of [7]). The next theorem will produce a measure density on $\mathscr{C}_t(\overline{U}))$ that will depend on the form α and that will be the main object of study in the first half of this article. Its product with the Wiener measure may be thought of as a generalized, or perturbed, Wiener measure; when $\alpha = 0$ it coincides with the usual Wiener measure.

THEOREM 2. *There exists a unique* $\rho_{\alpha,t}^{(U)} \in L^\infty(\mathscr{C}_t(\overline{U}))$ *with* $\|\rho_{\alpha,t}^{(U)}\|_{L^\infty(\mathscr{C}_t(\overline{U}))} \le 1$ *such that* $W_{\alpha,t}^{(U)}(f) = \int_{\mathscr{C}_t(\overline{U})} f \rho_{\alpha,t}^{(U)}\, \mathrm{d}w_t^{(U)}$ *for every* $f \in \mathrm{Cyl}(\mathscr{C}_t(\overline{U}))$.

3. A SEQUENCE OF APPROXIMATIONS FOR $\rho_{\alpha,t}^{(U)}$

So far, $\rho_{\alpha,t}^{(U)}$ has been constructed by a very abstract argument, therefore its various concrete properties are difficult to study. As a consequence, in what follows we shall construct a sequence of concrete approximations of this function, which will enjoy two essential properties: a group property, and the fact of being of absolute value 1. We shall then show that this sequence converges to $\rho_{\alpha,t}^{(U)}$ in $L^2(\mathscr{C}_t)$, so that these two properties will be transferred to $\rho_{\alpha,t}^{(U)}$, too. In order to complete this program, we shall now introduce several more ingredients.

Let P be a Borel regular probability on $[0,1]$; we shall see later on that the role of P will be to classify the various stochastic integrals that we shall obtain.

Whenever the points $x, y \in M$ may be joined by a unique minimizing geodesic, we shall denote it by $\gamma_{x,y} : [0,1] \to M$, where we understand that $\gamma(0) = x$ and $\gamma(1) = y$. Let us now define $I_P(\alpha) : M \times M \to \mathbb{R}$ by:

- $I_P(\alpha)(x,y) = \int_{[0,1]} \alpha_{\gamma_{x,y}(\tau)}(\dot\gamma_{x,y}(\tau))\, \mathrm{d}P(\tau)$, if there exists a unique minimizing geodesic $\gamma_{x,y}$ as above between x and y;

- $I_P(\alpha)(x,y) = 0$, otherwise.

For every $k \in \mathbb{N}$, let us now define the "approximations" $S_{P,t,k}(\alpha) : \mathscr{C}_t \to \mathbb{R}$ by

$$S_{P,t,k}(\alpha)(c) = \sum_{j=0}^{2^k-1} I_P(\alpha)\left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right) + \frac{t}{2^k}(\mathrm{d}^*\alpha)\left(c\left(\frac{jt}{2^k}\right)\right)\int_{[0,1]}(2\tau-1)\,\mathrm{d}P(\tau)\ .$$

So far, $\rho_{\alpha,t}^{(U)}$ has been obtained by a very abstract procedure (theorem 2), which makes its use in concrete calculations and the study of its properties very difficult. The following theorem remedies this situation, providing us with a concrete understanding of $\rho_{\alpha,t}^{(U)}$ as the limit of a sequence of functions given by explicit formulae.

THEOREM 3. $\lim_{k\to\infty} \mathrm{e}^{\mathrm{i}S_{P,t,k}(\alpha)}|_{\mathscr{C}_t(\overline{U})} = \rho_{\alpha,t}^{(U)}$ *in* $L^2(\mathscr{C}_t(\overline{U}), w_t^{(U)})$, *uniformly with respect to* t *in bounded subsets of* $(0,\infty)$, *and uniformly with respect to* $x_0 \in U$.

4. A UNITARY GROUP AND ITS GENERATOR

Let us now consider an exhaustion $M = \bigcup_{j\in\mathbb{N}} U_j$ of M with regular domains as the domain U used above (it exists as a consequence of proposition 2.28 in [8]). For notational simplicity, let us write $\rho_{\alpha,t}^{(j)}$ instead of $\rho_{\alpha,t}^{(U_j)}$, $h_\alpha^{(j)}$ instead of $h_\alpha^{(U_j)}$, and $w_t^{(j)}$ instead of $w_t^{(U_j)}$. So far we know that $\mathrm{e}^{\mathrm{i}S_{P,t,k}}|_{\mathscr{C}_t(\overline{U_j})} \to \rho_{\alpha,t}^{(j)}$ in $L^2(\mathscr{C}_t(\overline{U_j}), w_t^{(j)})$ for every $j \in \mathbb{N}$.

LEMMA 1. *The subset* $\mathscr{C}_t(\overline{U_j})$ *is closed in* $\mathscr{C}_t$ *for every* $j \geq 0$. *Similarly,* $\mathscr{C}_t(\overline{U_i})$ *is closed in* $\mathscr{C}_t(\overline{U_j})$ *for every* $i \leq j$.

The following lemma is as important as it is trivial.

LEMMA 2. *If* $i \leq j$ *then* $\rho_{\alpha,t}^{(j)}|_{\mathscr{C}_t(\overline{U_i})} = \rho_{\alpha,t}^{(i)}$ *almost everywhere on* $\mathscr{C}_t(\overline{U_i})$ *with respect to the measure* $w_t^{(i)}$.

The equality $\rho_{\alpha,t}^{(j)}|_{\mathscr{C}_t(\overline{U_i})} = \rho_{\alpha,t}^{(i)}$ almost everywhere for every $i \leq j$ implies the existence of the pointwise limit $\lim_{j\to\infty} \tilde{\rho}_{\alpha,t}^{(j)}$, which we shall denote by $\rho_{\alpha,t}$. It will be a measurable function (as pointwise limit of a sequence of measurable functions), and it will be bounded by 1 almost everywhere, because all the functions in the sequence are so. Therefore, it will be an element of $L^\infty(\mathscr{C}_t, w_t)$. Using the argument in the above lemma, one may show that $\rho_{\alpha,t}|_{\mathscr{C}_t(\overline{U_j})} = \rho_{\alpha,t}^{(j)}$ for every $j \geq 0$, as elements from $L^\infty(\mathscr{C}_t, w_t^{(j)})$.

After all these preliminary results, we may finally prove one of the core results of this work.

THEOREM 4.

$$\lim_{k\to\infty} \|\mathrm{e}^{\mathrm{i}S_{P,t,k}(\alpha)} - \rho_{\alpha,t}\|_{L^2(\mathscr{C}_t)} = 0$$

uniformly with respect to $t \in (0,T]$, *for every* $T > 0$.

COROLLARY 1. $\rho_{\alpha,t}$ *does not depend on the exhaustion with regular domains used.*

We have obtained that $\rho_{\alpha,t}$ is the limit of a sequence of exponentials with imaginary exponents. It is reasonable to ask whether $\rho_{\alpha,t}$ itself has such a form, and if the answer is affirmative to study its exponent. The answer to this question (and the moral justification of all the effort spent in obtaining all the technical results so far) is given by theorem 5.

THEOREM 5. *There exists a unique real-valued function* $\mathrm{Strat}_t(\alpha) \in L^0(\mathscr{C}_t)$ *such that* $\rho_{\alpha,t} = \mathrm{e}^{\mathrm{i}\,\mathrm{Strat}_t(\alpha)}$.

When we constructed the functions $S_{P,t,k}(\alpha)$, we did it in order for the functions $\mathrm{e}^{\mathrm{i}S_{P,t,k}(\alpha)}$ to approximate $\rho_{\alpha,t} = \mathrm{e}^{\mathrm{i}\,\mathrm{Strat}_t(\alpha)}$ in $L^2(\mathscr{C}_t)$. We shall see now that this approximation property extends, even though in a weaker form, to the exponents.

THEOREM 6. $\lim_{k\to\infty} S_{P,t,k}(\alpha) = \mathrm{Strat}_t(\alpha)$ *in measure, uniformly with respect to t in bounded subsets of* $(0,\infty)$.

We shall see in the next section, that Strat_t is the Stratonovich stochastic integral. The fact that it is the limit in measure of the sequence of approximations used above was already known; what is new is that it stems into existence as the generator of the unitary group considered above (or, giving up rigour, it is the "logarithm" of the function $\rho_{\alpha,t}$).

5. A GENERAL CONCEPT OF STOCHASTIC INTEGRAL

In order to unravel a general concept of stochastic integral, let us return to the approximations $S_{P,t,k}(\alpha)$ constructed above and define the related approximations

$$A_{P,t,k}(\alpha)(c) = \sum_{j=0}^{2^k-1} I_P(\alpha)\left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right) \tag{1}$$

for every curve $c \in \mathscr{C}_t$ (that is, we just drop the term containing $\mathrm{d}^*\alpha$). We shall now study the behaviour of these approximations on continuously differentiable curves, this "classical" behaviour going to guide us towards the understanding of its "stochastic" counterpart.

THEOREM 7. *If* $c : [0,t] \to M$ *is a twice continuously differentiable curve, then*

$$\int_c \alpha = \lim_{k\to\infty} A_{P,t,k}(\alpha)(c) \ .$$

We shall draw inspiration from the resemblance between theorem 7 and theorem 6 in order to exhibit a general concept of stochastic integral. Let $\mathrm{Prob}([0,1])$ the space of regular Borel probabilities on the real interval $[0,1]$.

Definition 1. We shall say that $\mathrm{Int}_t : \Omega^1(M) \to L^0(\mathscr{C}_t)$ is a stochastic integral if and only if there exists $P \in \mathrm{Prob}([0,1])$ such that $\mathrm{Int}_t(\alpha)$ be the limit in measure of the sequence of approximations $A_{P,t,k}(\alpha)$ for every $\alpha \in \Omega^1(M)$. When this condition is met we shall denote this stochastic integral by $\mathrm{Int}_{P,t}$, in order to emphasize its dependence on P.

Although the convergence in measure obtained in theorem 6 was uniform with respect to t from bounded subsets of $(0,\infty)$, we have not included this property in the above definition because it was not clear, upon writing this text, whether this uniformity is an essential ingredient of the concept or a merely accidental one without major consequences.

Remark 1. Given that convergence in measure (as in the proposed definition) is weaker than pointwise convergence, let us emphasize that $\mathrm{Int}_t(\alpha)(\cdot)$ must be understood not as a function defined for every curve from $\mathscr{C}_t$, but rather as an element from $L^0(\mathscr{C}_t)$. This is the major difference from the usual line integral, which is defined for every piecewise-differentiable curve.

Remark 2. It is worth noting, in light of theorem 7, that while the approximations $A_{P,t,k}$ evaluated along twice continuously differentiable curves converge to the same limit that does not depend on P, they converge in measure to different limits that do depend on P (more precisely, on its first-order moment, as we shall presently see).

Let $P \in \mathrm{Prob}([0,1])$. We would like to discover whether there exists any connection between the freshly defined $\mathrm{Int}_{P,t}(\alpha)$ and the function $\mathrm{Strat}_t(\alpha)$ obtained in theorem 5. Let us notice that

$$\lim_{k\to\infty} \frac{t}{2^k} \sum_{j=0}^{2^k-1} (\mathrm{d}^*\alpha)\left(c\left(\frac{jt}{2^k}\right)\right) = \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s$$

for every $c \in \mathscr{C}_t$, as the limit of the Riemann sums associated to the continuous function $(\mathrm{d}^*\alpha)\circ c$, the equidistant partition of $[0,t]$ into 2^k subintervals, and the intermediate points $\left(c(\frac{jt}{2^k})\right)_{0\le j\le 2^k-1}$. Even more, then, the above convergence is also valid in measure. If we pass to the limit in measure in formula (1) used to define the approximations $A_{P,t,k}$, we get

$$\mathrm{Int}_{P,t}(\alpha)(c) = \mathrm{Strat}_t(\alpha)(c) - \int_{[0,1]} (2\tau - 1)\,\mathrm{d}P(\tau) \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s\ ,$$

which shows that although the probability P may be extremely complicated, the corresponding stochastic integral $\mathrm{Int}_{P,t}$ remembers only its first-order moment, discarding any other information associated to P; furthermore, any two probabilities from $\mathrm{Prob}([0,1])$ with the same first order moment give rise to the same stochastic integral. We also conclude that, since the function Strat_t has already been constructed, $\mathrm{Int}_{P,t}$ exists for every $P \in \mathrm{Prob}([0,1])$. Since the function $2\tau - 1$ has minimum -1 and maximum 1 on $[0,1]$, and since P is a probability, it follows that $\int_{[0,1]}(2\tau-1)\,\mathrm{d}P(\tau) \in [-1,1]$, and that every stochastic integral Int_t on $\mathscr{C}_t$ is of the form $\mathrm{Int}_t(\alpha) = \mathrm{Strat}_t(\alpha) + \theta\int_0^t(\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s$ with $\theta \in [-1,1]$.

Furthermore, if $P, Q \in \mathrm{Prob}([0,1])$ then

$$\mathrm{Int}_{P,t}(\alpha)(c) = \mathrm{Int}_{Q,t}(\alpha)(c) - 2\int_{[0,1]} \tau\,\mathrm{d}(P-Q)(\tau) \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s\ ,$$

so that any two stochastic integrals differ by a multiple of the integral of $\mathrm{d}^*\alpha$.

This is a good moment two see several concrete examples of such stochastic integrals as defined in this work, and to compare our results to the ones already obtained in the stochastic literature.

- If $P = \delta_0$ (the Dirac measure concentrated at 0), then

$$\mathrm{Int}_{\delta_0,t}(\alpha)(c) = \mathrm{Strat}_t(\alpha)(c) + \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s\ .$$

By comparing our approximations $A_{\delta_0,k,t}(\alpha)$ of $I_{\delta_0,t}(\alpha)$ to the ones in theorem 7.37 on page 110 of [5] (or to the ones in theorem A from [2], which is nevertheless stated under more restrictive hypotheses than here), we recognize immediately that $\mathrm{Int}_{\delta_0,t}(\alpha)$ is the Itô integral of α, therefore from now on we shall denote it by $\mathrm{Ito}_t(\alpha)$.

- If $P = \mathrm{Leb}_{[0,1]}$ (the Lebesgue measure on $[0,1]$), or $P = \delta_{\frac{1}{2}}$ (the Dirac measure concentrated at $\frac{1}{2}$), or $P = \frac{1}{2}\delta_1$, or $P = \frac{1}{2}(\delta_0 + \delta_1)$, then the corresponding stochastic integral is

$$\mathrm{Int}_{\mathrm{Leb}_{[0,1]},t}(\alpha) = \mathrm{Strat}_t(\alpha)\ .$$

By comparing the approximations $A_{\mathrm{Leb}_{[0,1]},k,t}(\alpha)$ of $\mathrm{Int}_{\mathrm{Leb}_{[0,1]},t}(\alpha)$ to those in theorem 7.14 on page 96 of [5], we readily recognize that $\mathrm{Int}_{\mathrm{Leb}_{[0,1]},t}(\alpha)$ is the Stratonovich integral of α. (The reader is invited to compare these results to the ones in section 6 of [12], too.)

- In general, if $M_1(P) = \int_{[0,1]} \tau\,\mathrm{d}P(\tau)$, then the stochastic integral $\mathrm{Int}_{P,t}(\alpha)$ corresponding to $P \in \mathrm{Prob}([0,1])$ coincides with the one produced by the probabilities $\delta_{M_1(P)}$ (the Dirac measure concentrated at $M_1(P) \in [0,1]$) and $(1 - M_1(P))\delta_0 + M_1(P)\delta_1$, all these probabilities having $M_1(P)$ as first order moment. Nevertheless, although in principle we could study the stochastic integrals defined in this work using only these very simple combinations of Dirac measures, some results are much easier to prove using more complicated probabilities with the same first order moment. In particular, in the study of the Stratonovich integral it is usually more convenient to use the Lebesgue measure on $[0,1]$.

Remark 3. The previous examples show that the Stratonovich and Itô integrals of α are equal if and only if $\mathrm{d}^* \alpha = 0$. This is worth comparing to lemma 8.24 on page 120 of [5], where only a necessary but not sufficient condition (difficult to verify in concrete applications) is given that guarantees this equality. More specifically, Émery first introduces the concept of stochastic parallel transport in the bundles TM and in T^*M, starting from which he constructs certain martingales depending on α; if these martingales are of finite variation, then the Stratonovich and Itô integrals of α are equal.

As always when one studies objects that depend on certain parameters, it is useful to study how regular this dependency is. In particular, it is interesting to study the dependence of the stochastic integral $\mathrm{Int}_{P,t}(\alpha)$ on the parameter $t \in (0,T]$, where $T > 0$ is arbitrary. Since the stochastic integral $\mathrm{Int}_{P,t}(\alpha)$ lives in the space $L^0(\mathscr{C}_t)$ for each $t \in (0,T]$, and since all these spaces are unrelated to each other, we shall have to embed all of them in the bigger space $L^0(\mathscr{C}_T)$. In order to do this, let us remember that the natural topology in $L^0(\mathscr{C}_t)$ is that of convergence in the Wiener measure w_t. If $\mathrm{res}_{[0,t]} : \mathscr{C}_T \to \mathscr{C}_t$ is the restriction $\mathrm{res}_{[0,t]}(c) = c|_[0,t]$, then clearly $w_t = (\mathrm{res}_{[0,t]})_* w_T$.

For the line integral, if $c : [0,T] \to M$ is continuously-differentiable, then

$$\left| \int_c \alpha - \int_{\mathrm{res}_{[0,t]}(c)} \alpha \right| = \left| \int_{\mathrm{res}_{[t,T]}(c)} \alpha \right| \leq \sup_{s \in [0,T]} |\alpha_{c(s)}(\dot{c}(s))| \, (T-t) \, .$$

In particular, the map $[0,T] \ni t \mapsto \int_{\mathrm{res}_{[0,t]}(c)} \alpha \in \mathbb{R}$ is continuous. The following theorem offers a weaker analogue of this fact in the context of stochastic integration.

THEOREM 8. *For every $\alpha \in \Omega^1(M)$, the map $(0,T] \ni t \mapsto \mathrm{Int}_{P,t}(\alpha) \circ \mathrm{res}_{[0,t]} \in L^0(\mathscr{C}_T)$ is continuous.*

6. CONCLUSION AND ACKNOWLEDGEMENTS

The main aim has been to show how to give an alternative construction of some basic objects in stochastic analysis using only functional-analytic tools, without it being necessary to resort to probability-theoretical concepts or techniques. Another aim has been to advance a point of view allowing the entire subject of stochastic integration to be seen unravelling from a small number of fundamental ideas, along lines emphasizing the deep analogies with curvilinear integration. The strategy adopted herein has allowed the classification of stochastic integrals and the exhibition of the simple relationship connecting any two of them.

From a technical point of view, since all the objects involved were intrinsic to the manifold M, we have obtained that their construction be intrinsic, too. This differs from the approach that other stochastic analysis on manifolds textbooks use (for instance [6]), which resort to embedding the underlying manifold in Euclidean spaces using Whitney's theorem, thus using extrinsic geometrical tools to obtain intrinsic results. We have also attempted to keep the prerequisites to a minimum, using only a handful of basic functional-analytic tools. Once the foundations of this construction are laid down, developing the various properties of stochastic integrals becomes much easier than in the traditional probability-theoretic textbooks. We have thus not needed to use Cartan's rolling map, as it is done in [4]. Neither has it been necessary to choose an interpolation rule, as done in [5] (which requires the use of the measurable selection theorem, checking the hypotheses of which further requires working with the Whitney topology on the space of smooth curves in M), its role being taken on by the cut-off function χ as well as by the truncation by 0 of the expression $\int_{[0,1]} \alpha_{\gamma_{x,y}(\tau)}(\dot{\gamma}_{x,y}(\tau)) \, \mathrm{d}P(\tau)$ for y far away from x. Unlike in [3], M is not required to be compact. We have also not needed to work with second order tangent vectors and Laurent Schwartz' second order differential geometry, as done by Émery. This parsimonious use of fundamental concepts and technical means has been one of the driving goals of the present text which is built upon the belief that conceptual and technical minimality must be an imperative of any intellectual construction.

The key points to remember from this article are:

- the approximations $c \mapsto \sum_{j=0}^{2^k-1} I_P(\alpha)\left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right)$ converge in measure to a limit denoted $\mathrm{Int}_{P,t}(\alpha)$, for every regular Borel probability P on the interval $[0,1]$; called "the stochastic integral corresponding to P";

- $\mathrm{Int}_{\delta_0,t}$ is the Itô integral and if $P = \mathrm{Leb}_{[0,1]}$, or $P = \delta_{\frac{1}{2}}$, or $P = \frac{1}{2}(\delta_0 + \delta_1)$, then the corresponding stochastic integral $\mathrm{Int}_{P,t}$ is the Stratonovich integral;

- for any probabilities P and Q as above, $\mathrm{Int}_{P,t}(\alpha)(c) = \mathrm{Int}_{Q,t}(\alpha)(c) - 2\int_{[0,1]} \tau \,\mathrm{d}(P-Q)(\tau) \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s$.

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