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SOME CHARACTERIZATIONS OF LIPSCHITZ SPACES VIA COMMUTATORS OF THE HARDY-LITTLEWOOD MAXIMAL OPERATOR ON SLICE SPACES

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Abstract. Let M be the Hardy-Littlewood maximal operator and b be a locally integrable function. Denote by M_b and $[b,M]$ the maximal commutator and the nonlinear commutator of M with b. In this paper, we give necessary and sufficient conditions for the boundedness of M_b and $[b,M]$ on slice spaces when the function b belongs to Lipschitz spaces, by which a new characterization of non-negative Lipschitz functions is obtained.

***Key words**:* slice space, Lipschitz space, Hardy-Littlewood maximal operator, commutator.
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1. INTRODUCTION AND MAIN RESULTS

Let T be the classical singular integral operator and b the be locally integrable function, the commutator $[b,T]$ is defined by

$$[b,T]f(x) = bTf(x) - T(bf)(x).$$

In 1976, Coifman, Rochberg and Weiss [5] stated that the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$. The bounded mean oscillation space $BMO(\mathbb{R}^n)$ was introduced by John and Nirenberg [10], which is defined as the set of all locally integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, \mathrm{d}x < \infty,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and $f_Q := \frac{1}{|Q|} \int_Q f(x) \mathrm{d}x$. In 1978, Janson [8] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$(see Definition 1) via the commutator $[b,T]$ and proved that $[b,T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)\,(0 < \beta < 1)$, where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also Paluszyński [15]).

As usual, a cube $Q \subset \mathbb{R}^n$ always means its sides parallel to the coordinate axes. Denote by $|Q|$ the Lebesgue measure of Q and χ_Q the characteristic function of Q. For $1 \le p \le \infty$, we denote by p' the conjugate index of p, namely, $p' = p/(p-1)$. We always denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $f \lesssim g$ means that $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$.

For a locally integrable function f, the Hardy-Littlewood maximal operator M is given by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The maximal commutator of M with a locally integrable function b is defined by

$$M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)| \, \mathrm{d}y,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x.

The mapping properties of the maximal commutator M_b have been studied intensively by many authors, we refer the readers to see [1, 7, 16–18, 21] and therein references. The following two results can be found in [7] and [21].

THEOREM 1 ([7]). *Let $1 < p < \infty$ and b be a locally integrable function. Then the maximal commutator M_b is bounded on $L^p(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

THEOREM 2 ([21]). *Let $0 < \beta < 1$ and b be a locally integrable function. If $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$, then the commutator M_b is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.*

The first part of this paper is to study the boundedness of M_b when the function b belongs to Lipschitz spaces, a characterization of Lipschitz spaces via such commutator is given.

Definition 1. Let $0 < \beta < 1$, we say a function b belongs to the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$,

$$|b(x) - b(y)| \le C|x - y|^\beta.$$

The smallest such constant C is called the $\dot{\Lambda}_\beta$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_\beta}$.

In 2019, Auscher and Mourgoglou [2] introduced the slice space $(E_2^p)_t(\mathbb{R}^n)$ with $0 < t < \infty$ and $1 < p < \infty$, they studied the weak solutions of boundary value problems with a t-independent elliptic systems in the upper half plane. Recently, Auscher and Prisuelos-Arribas [3] studied the boundedness of some classical operators on the slice space $(E_r^p)_t(\mathbb{R}^n)$ with $0 < t < \infty$ and $1 < p, r < \infty$, where these operators include the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator with the standard kernel, the Riesz potential and the Riesz transform associated with the second order divergence form elliptic operator.

For $0 < p < \infty$, the Lebesgue space $L^p(\mathbb{R}^n)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} := \left[\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x \right]^{\frac{1}{p}} < \infty.$$

Definition 2. Let $0 < t < \infty$ and $1 < r, p < \infty$. The slice space $(E_r^p)_t(\mathbb{R}^n)$ is defined as the set of all locally r-integrable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{(E_r^p)_t(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \left(\frac{1}{|Q(x,t)|} \int_{Q(x,t)} |f(y)|^r \mathrm{d}y \right)^{\frac{p}{r}} \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

If we take $r = p$, then the slice space $(E_r^p)_t(\mathbb{R}^n)$ is the Lebesgue space $L^p(\mathbb{R}^n)$. For a cube Q, we denote by $\|f\|_{(E_r^p)_t(Q)} = \|f\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)}$.

Our first result can be stated as follows.

THEOREM 3. *Let $0 < \beta < 1$, $0 < t < \infty$ and b be a locally integrable function. If $1 < p < r < \infty$, $1 < q < s < \infty$ and $\beta/n = 1/p - 1/r = 1/q - 1/s$, then the following statements are equivalent:*

(1) *$b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.*

(2) *M_b is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.*

(3) *There exists a constant $C > 0$ such that*

$$\sup_Q \frac{1}{|Q|^{\beta/n + 1/s}} \|b(\cdot) - b_Q\|_{(E_r^s)_t(Q)} \le C. \tag{1.1}$$

(4) *There exists a constant* $C>0$ *such that*

$$\sup_{Q}\frac{1}{|Q|^{1+\beta/n}}\int_{Q}|b(x)-b_{Q}|\,\mathrm{d}x\le C. \tag{1.2}$$

For a locally integrable function f, the nonlinear commutator of the Hardy-Littlewood maximal operator M with a function b is defined by

$$[b,M]f(x)=bMf(x)-M(bf)(x).$$

Using real interpolation techniques, Milman and Schonbek [14] obtained the commutator $[b,M]$ is bounded on $L^p(\mathbb{R}^n)$, when $b\in BMO(\mathbb{R}^n)$ and $b\ge 0$. In 2000, Bastero, Milman and Ruiz [4] studied the necessary and sufficient conditions for the boundedness of $[b,M]$ on the Lebesgue space $L^p(\mathbb{R}^n)$ for $1<p<\infty$. In 2014, Zhang and Wu [23] obtained similar results and extended the mentioned results to variable exponent Lebesgue spaces.

We would like to remark that the commutators M_b and $[b,M]$ essentially differ from each other. For example, M_b is positive and sublinear, but $[b,M]$ is neither positive nor sublinear.

The second part of this paper aims to study the mapping properties of the nonlinear commutator $[b,M]$ when the function b belongs to Lipschitz spaces and $b\ge 0$. To state our results, we recall the definition of the maximal operator with respect to a cube. For a fixed cube Q_0, the Hardy-Littlewood maximal function with respect to Q_0 of a function f is given by

$$M_{Q_0}(f)(x)=\sup_{Q_0\supseteq Q\ni x}\frac{1}{|Q|}\int_{Q}|f(y)|\mathrm{d}y,$$

where the supremum is taken over all the cubes Q with $Q\subseteq Q_0$ and $Q\ni x$.

The mapping properties of the nonlinear commutator $[b,M]$ have been investigated widely, we refer the readers to see [11, 19–22] and therein references. Zhang [21] obtained that the following result, which is a characterization of non-negative Lipschitz functions on the Lebesgue space $L^p(\mathbb{R}^n)$.

THEOREM 4 ([21]). *Let* $0<\beta<1$ *and* b *be a locally integrable function. If* $1<p<n/\beta$ *and* $1/q=1/p-\beta/n$*, then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ *and* $b\ge 0$*;*

(2) $[b,M]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$*;*

(3) *there exists a constant* $C>0$ *such that*

$$\sup_{Q}\frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_{Q}|b(x)-M_{Q}(b)(x)|^{q}\,\mathrm{d}x\right)^{1/q}\le C.$$

Our second result can be stated as follows.

THEOREM 5. *Let* $0<\beta<1$, $0<t<\infty$ *and* b *be a locally integrable function. If* $1<p<r<\infty$, $1<q<s<\infty$ *and* $\beta/n=1/p-1/r=1/q-1/s$*, then the following statements are equivalent:*

(1) $b\in\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ *and* $b\ge 0$.

(2) $[b,M]$ *is bounded from* $(E_p^q)_t(\mathbb{R}^n)$ *to* $(E_r^s)_t(\mathbb{R}^n)$.

(3) *There exists a constant* $C>0$ *such that*

$$\sup_{Q}\frac{1}{|Q|^{\beta/n+1/s}}\|b(\cdot)-M_{Q}(b)(\cdot)\|_{(E_r^s)_t(Q)}\le C. \tag{1.3}$$

(4) *There exists a constant* $C>0$ *such that*

$$\sup_{Q}\frac{1}{|Q|^{1+\beta/n}}\int_{Q}|b(x)-M_{Q}(b)(x)|\,dx\le C. \tag{1.4}$$

2. PRELIMINARIES

To prove our results, we need some necessary lemmas. It is known that the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ coincides with some Morrey-Companato space (see [9] for example) and can be characterized by mean oscillation as the following lemma, which is due to DeVore and Sharpley [6] and Janson, Taibleson and Weiss [9] (see also Paluszyński [15]).

LEMMA 1. *Let $0<\beta<1$ and $1\le q<\infty$. The space $\dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f such that*

$$\|f\|_{\dot{\Lambda}_{\beta,q}} = \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^q\,\mathrm{d}x\right)^{1/q} < \infty.$$

Then, for all $0<\beta<1$ and $1\le q<\infty, \dot{\Lambda}_\beta(\mathbb{R}^n)=\dot{\Lambda}_{\beta,q}(\mathbb{R}^n)$ with equivalent norms.

Let $0<\alpha<n$ and f be a locally integrable function, the fractional maximal function of f is given by

$$M_\alpha(f)(x) = \sup_Q \frac{1}{|Q|^{1-\alpha/n}}\int_Q |f(y)|\mathrm{d}y,$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ containing x.

The following lemma is given by Lu, Wang and Zhou [12], they obtained that the boundedness of the fractional maximal operator M_α on slice spaces.

LEMMA 2. *Let $0<t<\infty$, $1<p<r<\infty$ and $1<q<s<\infty$ with $\alpha/n=1/p-1/r=1/q-1/s$ for $0<\alpha<n$. If $f\in(E_p^q)_t(\mathbb{R}^n)$, then*

$$\|M_\alpha f\|_{(E_r^s)_t(\mathbb{R}^n)} \le C\|f\|_{(E_p^q)_t(\mathbb{R}^n)},$$

where the positive constant C is independent of f and t.

LEMMA 3 [13]. *Let $0<t<\infty$, $1<p,r<\infty$ and Q be a cube in $\mathbb{R}^n$. Then*

$$\|\chi_Q\|_{(E_r^p)_t(\mathbb{R}^n)} \sim |Q|^{1/p},$$

LEMMA 4 [4]. *For any fixed cube Q, let $E=\{x\in Q: b(x)\le b_Q\}$ and $F=\{x\in Q: b(x)>b_Q\}$. Then the following equality is true:*

$$\int_E |b(x)-b_Q|\,\mathrm{d}x = \int_F |b(x)-b_Q|\,\mathrm{d}x.$$

3. PROOFS OF THEOREMS 3 AND 5

Proof of Theorem 3. (1) $\Rightarrow$ (2): Assume $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$. For any fixed cube $Q\subset\mathbb{R}^n$, we have

$$\begin{aligned} M_b(f)(x) &= \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |b(x)-b(y)||f(y)|\,\mathrm{d}y \\ &\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \sup_{Q\ni x}\frac{1}{|Q|^{1-\beta/n}}\int_Q |f(y)|\,\mathrm{d}y \\ &= C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} M_\beta f(x). \end{aligned}$$

By Lemma 2, we obtain that M_b is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(2) $\Rightarrow$ (3): For any fixed cube $Q \subset \mathbb{R}^n$ and all $x \in Q$, we have

$$\begin{aligned}|b(x)-b_Q| &\leq \frac{1}{|Q|}\int_Q |b(x)-b(y)|\,\mathrm{d}y\\ &= \frac{1}{|Q|}\int_Q |b(x)-b(y)|\chi_Q(y)\,\mathrm{d}y\\ &\leq M_b(\chi_Q)(x).\end{aligned}$$

Since M_b is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$, then by Lemma 3 and noting that $\beta/n = 1/q - 1/s$, we obtain that

$$\begin{aligned}\frac{1}{|Q|^{\beta/n+1/s}}\|b(\cdot)-b_Q\|_{(E_r^s)_t(Q)} &\leq \frac{1}{|Q|^{\beta/n+1/s}}\|M_b(\chi_Q)(\cdot)\|_{(E_r^s)_t(Q)}\\ &\leq \frac{C}{|Q|^{\beta/n+1/s}}\|\chi_Q\|_{(E_p^q)_t(\mathbb{R}^n)}\\ &\leq C,\end{aligned}$$

which implies (1.1) since the cube $Q \subset \mathbb{R}^n$ is arbitrary.

(3) $\Rightarrow$ (4): Assume (1.1) holds, we will prove (1.2). For any fixed cube Q, by Hölder's inequality and Lemma 3, it is easy to see that

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\,\mathrm{d}x &\leq \frac{C}{|Q|^{1+\beta/n}}\|b(\cdot)-b_Q\|_{(E_r^s)_t(Q)}\|\chi_Q\|_{(E_{r'}^{s'})_t(\mathbb{R}^n)}\\ &\leq \frac{C}{|Q|^{\beta/n+1/s}}\|b(\cdot)-b_Q\|_{(E_r^s)_t(Q)}\\ &\leq C.\end{aligned}$$

(4) $\Rightarrow$ (1): It follows from Lemma 1 directly, thus we omit the details.

The proof of Theorem 3 is completed. □

Proof of Theorem 5. (1) $\Rightarrow$ (2): Assume $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ and $b \geq 0$. For any locally integral function f, we have

$$\begin{aligned}|[b,M](f)(x)| &= |b(x)M(f)(x) - M(bf)(x)|\\ &\leq \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |b(x)-b(y)||f(y)|\,\mathrm{d}y\\ &\leq C\|b\|_{\dot{\Lambda}_\beta}\sup_{Q\ni x}\frac{1}{|Q|^{1-\beta/n}}\int_Q |f(y)|\,\mathrm{d}y\\ &\leq C\|b\|_{\dot{\Lambda}_\beta}M_\beta(f)(x).\end{aligned}$$

By Lemma 2, we obtain that $[b,M]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$.

(2) $\Rightarrow$ (3):For any fixed cube Q and any $x \in Q$, it is easy to see that

$$M_Q(\chi_Q)(x) = \chi_Q(x) \text{ for any } x \in Q,$$

then we have

$$M(\chi_Q)(x) = \chi_Q(x) \text{ and } M(b\chi_Q)(x) = M_Q(b)(x) \text{ for any } x \in Q. \tag{3.1}$$

By (3.1), we obtain that

$$b(x) - M_Q(b)(x) = b(x)M(\chi_Q)(x) - M(b\chi_Q)(x) = [b,M](\chi_Q)(x).$$

Since $[b,M]$ is bounded from $(E_p^q)_t(\mathbb{R}^n)$ to $(E_r^s)_t(\mathbb{R}^n)$ and $\beta/n = 1/q - 1/s$, by Lemma 3, we have

$$\begin{aligned}\frac{1}{|Q|^{\beta/n+1/s}}\|b(\cdot)-M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)} &= \frac{1}{|Q|^{\beta/n+1/s}}\|[b,M](\chi_Q)(\cdot)\|_{(E_r^s)_t(Q)}\\ &\le \frac{C}{|Q|^{\beta/n+1/s}}\|\chi_Q\|_{(E_p^q)_t(\mathbb{R}^n)}\\ &\le C,\end{aligned}$$

which implies (1.3).

(3) ⇒ (4): Assume (1.3) holds, then for any fixed cube Q, by Hölder's inequality and (1.3), it is easy to see that

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\,\mathrm{d}x &\le \frac{C}{|Q|^{1+\beta/n}}\|b(\cdot)-M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)}\|\chi_Q\|_{(E_{r'}^{s'})_t(\mathbb{R}^n)}\\ &\le \frac{C}{|Q|^{\beta/n+1/s}}\|b(\cdot)-M_Q(b)(\cdot)\|_{(E_r^s)_t(Q)}\\ &\le C,\end{aligned}$$

where the constant C is independent of Q. Thus we have (1.4).

(4) ⇒ (1): To prove $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, by Lemma 1, it suffices to show that there is a constant $C > 0$ such that for any fixed cube Q,

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\,\mathrm{d}x \le C.$$

For any fixed cube Q, let $E = \{x \in Q : b(x) \le b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$. Since for any $x \in E$ we have $b(x) \le b_Q \le M_Q(b)(x)$, then for any $x \in E$,

$$|b(x)-b_Q| \le |b(x)-M_Q(b)(x)|.$$

By Lemma 4 and (1.4), we have

$$\begin{aligned}\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|\,\mathrm{d}x &= \frac{1}{|Q|^{1+\beta/n}}\int_{E\cup F} |b(x)-b_Q|\,\mathrm{d}x\\ &= \frac{2}{|Q|^{1+\beta/n}}\int_E |b(x)-b_Q|\,\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_E |b(x)-M_Q(b)(x)|\,\mathrm{d}x\\ &\le \frac{2}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\,\mathrm{d}x\\ &\le C.\end{aligned}$$

Thus we obtain that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$. Next, we will prove $b \ge 0$, it suffices to show $b^- = 0$, where $b^- = -\min\{b,0\}$. Let $b^+ = |b| - b^-$, then $b = b^+ - b^-$. For any fixed cube Q and $x \in Q$, we observe that

$$0 \le b^+(x) \le |b(x)| \le M_Q(b)(x),$$

then we have

$$0 \le b^-(x) \le M_Q(b)(x) - b^+(x) + b^-(x) = M_Q(b)(x) - b(x).$$

Combining with the above estimates and (1.4), we can see that

$$\begin{aligned}\frac{1}{|Q|}\int_Q b^-(x)\mathrm{d}x &\le \frac{1}{|Q|}\int_Q |M_Q(b)(x)-b(x)|\\ &\le |Q|^{\beta/n}\left(\frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-M_Q(b)(x)|\,\mathrm{d}x\right)\\ &\le C|Q|^{\beta/n}.\end{aligned}$$

Thus, $b^-=0$ follows from Lebesgue's differentiation theorem.

The proof of Theorem 5 is completed. □

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