ESTIMATING \( \text{PROB}\{Y < X\} \) IN THE CASE OF THE POWER DISTRIBUTION

Ion VLADIMIRESCU, Adrian IAŞINSCHI

Fac. Mathematics-Informatics, University of Craiova
Corresponding author: Ion VLADIMIRESCU, E-mail: vladi@central.ucv.ro

We consider the problem of estimating the probability \( \text{Prob}\{Y < X\} \), where \( X \) and \( Y \) are two independent random variables having power distributions. We obtain a parametric estimator \( \hat{R}_n \) and a non-parametric estimator \( \overline{R}_n \) for the quantity \( R = \text{Prob}\{Y < X\} \). We compare the performances of these two estimators using Monte Carlo techniques and we find that the procedure used for the estimates is satisfactory.

AMS 2000 Subject Classification: 62N05, 62N02, 62G05.
Key words: stress-strength model; mechanical reliability; parametric and non-parametric estimates; Monte Carlo simulation.

1. INTRODUCTION

The problem of estimating \( R = \text{Prob}\{Y < X\} \) where \( X \) and \( Y \) are independent random variables has been studied for the exponential distribution by Kelley et al. [3], for the double exponential distribution in Awad and Fayoumi [2], and for the Lučenč distribution [5]. We shall consider the power distribution \( \pi_{b,\delta} \) with parameters \( b, \delta > 0 \), that is the distribution on \( \mathbb{R} \) with the probability density function:

\[
\rho(x;b,\delta) = \delta b^{-\delta} x^{\delta-1} 1_{(0,b)}(x), \quad x \in \mathbb{R}
\]

Let \( X \) and \( Y \) be two independent random variables such that \( X \sim \pi_{b_1,\delta_1} \), \( Y \sim \pi_{b_2,\delta_2} \). We can think of \( X \) as the strength of a mechanical system being subjected to a stress \( Y \). The purpose of this paper is to give a measure of the mechanical reliability of the system, that is estimating \( \text{Prob}\{Y < X\} \) with \( \delta_1, \delta_2 \) known and \( b_1, b_2 \) unknown.

2. ESTIMATING THE RELIABILITY

Since

\[
\text{Prob}(Y < X \mid X) = \begin{cases} \left( \frac{X}{b_2} \right)^{\delta_2}, & \text{when } 0 < X < b_2, \\ 1, & \text{when } X \geq b_2 \end{cases}
\]

and

\[
\text{Prob}(Y < X) = E(\text{Prob}(Y < X \mid X))
\]

we obtain
Ion Vladimirescu and Adrian Iaşinschi

\[
\text{Prob}(Y < X) = \int_0^\infty \text{Prob}(Y < X | X = x)p(x; b_1, \delta_1)dx = \begin{cases}
\int_0^{b_1} \frac{x^{\delta_2}}{b_1^{\delta_2}} \delta_1 b_1^{-\delta_1} x^{\delta_1 - 1} dx, & \text{if } b_1 \leq b_2, \\
\int_0^{b_2} \frac{x^{\delta_2}}{b_2^{\delta_2}} \delta_1 b_2^{-\delta_1} x^{\delta_1 - 1} dx + \int_{b_1}^{b_2} \frac{x^{\delta_2}}{b_2^{\delta_2}} \delta_1 b_1^{-\delta_1} x^{\delta_1 - 1} dx, & \text{if } b_1 > b_2
\end{cases}
\]

Let \( t = \frac{b_1}{b_2} \). Then we define

\[
f(t) = \text{Prob}(Y < X) = \begin{cases}
t^{\delta_2} \frac{\delta_1}{\delta_1 + \delta_2}, & \text{if } 0 < t \leq 1, \\
1 - t^{-\delta_2} \frac{\delta_2}{\delta_1 + \delta_2}, & \text{if } t > 1
\end{cases}
\]

The function \( f(t) \) is obviously continuous and differentiable on \((0, \infty)\), increasing and convex on \((0,1)\) and non-convex on \((1, \infty)\), as we can see in the following:

\[
f'(t) = \begin{cases}
\frac{\delta_1 \delta_2}{\delta_1 + \delta_2} t^{\delta_2 - 1}, & \text{if } 0 < t \leq 1, \\
\frac{\delta_1 \delta_2}{\delta_1 + \delta_2} (1 - t)^{-\delta_2 - 1}, & \text{if } t > 1,
\end{cases} \quad f''(t) = \begin{cases}
\frac{\delta_1 \delta_2 (\delta_2 - 1)}{\delta_1 + \delta_2} t^{\delta_2 - 2}, & \text{if } 0 < t \leq 1, \\
\frac{-\delta_1 \delta_2 (\delta_1 + 1)}{\delta_1 + \delta_2} t^{-\delta_2 - 2}, & \text{if } t > 1
\end{cases}
\]

The graph of \( f \) has the following appearance:

For fixed values of the parameters \( \delta_1, \delta_2 \) we can obtain the value of \( t \) necessary to achieve a reliability value of 0.95, or 0.05, specifically \( t_{0.95} = 2.9240 \), \( t_{0.05} = 0.0444 \), for \( \delta_1 = 1.5, \delta_2 = 0.5 \). In the case \( \delta_1 = \delta_2 \) we have \( t_{1-\alpha} = \frac{1}{t_{\alpha}} \).

3. ESTIMATION OF \( R = \text{Prob}(Y < X) \)

**Remark.** Let \( Z \sim \pi_{b, \delta} \) and \( z_1, \ldots, z_n \) be a random sample from \( Z \). Since \( E(Z) = \frac{b \delta}{\delta + 1} \), for \( \delta > 0 \) known, with the method of moments we find that
is an estimate for the parameter $b$.

Obviously, $\hat{b}_n = \varphi(z_n)$, where $\varphi:(0,\infty) \to (0,\infty), \varphi(t) = \frac{1+\delta}{\delta} t$, is a differentiable function. Then, (see [4], pg. 119), it follows that

$$\hat{b}_n \xrightarrow{a.s.} b, \text{ as } n \to \infty \quad (2)$$

Now let $x_1,\ldots,x_n$ be a random sample from $X$ and $y_1,\ldots,y_n$ a random sample from $Y$. We assume that the samples are independent. Taking into account the fact that

$$\frac{1}{\delta_1} < \frac{1}{\delta_2} \quad (3)$$

and using the method of moments, we obtain that

$$\hat{b}_{1,n} = \frac{1+\delta_1}{\delta_1} \bar{x}_n \quad \text{and} \quad \hat{b}_{2,n} = \frac{1+\delta_2}{\delta_2} \bar{y}_n$$

are estimates for the parameters $b_1$ and $b_2$.

From (2) it follows that

$$\hat{b}_{1,n} \xrightarrow{a.s.} b_1 \quad \text{and} \quad \hat{b}_{2,n} \xrightarrow{a.s.} b_2 \quad \text{as } n \to \infty.$$ 

Then,

$$\frac{\hat{b}_{1,n}}{\hat{b}_{2,n}} \xrightarrow{a.s.} \frac{b_1}{b_2}, \text{ as } n \to \infty \quad (3)$$

Since $f$ is continuous, from (3) we get

$$\hat{R}_n \overset{\text{def}}{=} f\left(\frac{\hat{b}_{1,n}}{\hat{b}_{2,n}}\right) \xrightarrow{a.s.} f\left(\frac{b_1}{b_2}\right) \text{ as } n \to \infty. \quad (4)$$

Therefore, for $n$ large enough,

$$f\left(\frac{b_1}{b_2}\right) = \hat{R}_n,$$

that is

$$\hat{R}_n = f\left(\frac{\hat{b}_{1,n}}{\hat{b}_{2,n}}\right) = \begin{cases} \frac{\delta_1}{\delta_1 + \delta_2} \left(\frac{\hat{b}_{1,n}}{\hat{b}_{2,n}}\right)^{\delta_2}, & \text{if } \hat{b}_{1,n} \leq \hat{b}_{2,n} \quad (\text{if } \hat{r}_n \leq 1) \\ \frac{\delta_1}{\delta_1 + \delta_2} \left(\frac{\hat{b}_{1,n}}{\hat{b}_{2,n}}\right)^{-\delta_1}, & \text{if } \hat{b}_{1,n} > \hat{b}_{2,n} \quad (\text{if } \hat{r}_n > 1) \end{cases}$$

where $\hat{r}_n = \frac{\hat{b}_{1,n}}{\hat{b}_{2,n}}$, is an estimate for $R = f\left(\frac{b_1}{b_2}\right) = \text{Prob}(Y < X)$. 

---

Estimating Prob $[Y<X]$ in the case of the power distribution
4. SIMULATION STUDY

In this section we will consider, besides the parametric estimator \( \hat{R}_n \), defined in the preceding section, a non-parametric estimator (see [5]), defined as follows:

\[
\bar{R}_n = \frac{\text{card}\{(X_i, Y_j) \mid Y_j < X_i, \ 1 \leq i, j \leq n\}}{n^2}
\]

We will compare the mean bias (MB) and mean square error (MSE) for the two estimators. For an estimator \( R \) we define the two above quantities by means of the following formulas:

\[
\text{MB}(R) = \frac{1}{N} \sum_{i=1}^{N} (R(r_i; \delta_1, \delta_2) - R(r_i; \delta_i, \delta_i))
\]

\[
\text{MSE}(R) = \frac{1}{N} \sum_{i=1}^{N} (R(r_i; \delta_1, \delta_2) - R(r_i; \delta_i, \delta_i))^2
\]

where \( N \) represents the number of experiments, in our case estimating \( R \).

Random samples from \( X \sim \pi_{b_1, \delta_1} \), \( Y \sim \pi_{b_2, \delta_2} \) were generated, with \((b_1, b_2) \in \{(3,4), (3,6), (3,10), (3,10)\}\) and \((\delta_1, \delta_2) \in \{(0.5,0.5), (0.5,1.5), (0.5,5), (0.5,10)\}\). In order to obtain the MB and MSE the experiment was repeated \( N = 1000 \) times. The results can be obtained, on request, from the authors. The simulation showed that:

1) \( \hat{R}_n \) and \( \bar{R}_n \) estimate \( R \) with errors of the \( 10^{-2} \) order in the worse case, that is when \( b_1 = b_2 \).

According to the values obtained for the MSE, \( \hat{R}_n \) is superior. Also we noticed that MSE appears to decrease exponentially when the sample size increases, as in the following plots (MSE versus sample size):
2) Generally, $\hat{R}_n$ underestimates $R$, as in the next plots ($MB$ versus sample size):
3) $MSE(\hat{R}_n)$ increases when $r = \frac{b_1}{b_2}$ is approaching 1, as it can be observed in the next plot ($MSE$ versus $r$):

![Plot showing $MSE$ vs $r$]

We conclude that both estimators appear to work well, with an advantage for the parametric estimator $\hat{R}_n$.

**Acknowledgement.** We would like to express our gratitude to Prof. Dr. Viorel Gh. Voda for his useful observations and suggestions.

**REFERENCES**


*Received July 28, 2004*