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CONSIDERATIONS ON THE VIRTUAL MECHANICAL WORK PRINCIPLE

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The systems of critical material particles are defined as systems where the rank of the functional matrix decreases when Cartesian coordinates are replaced with the generalized coordinates. Proof is further provided that the virtual mechanical work principle is a necessary and sufficient condition only in case of non-critical material particle systems.

We will consider an orthogonal Cartesian system of Oxyz coordinates and a mechanical system of N particles $A_i(x_i, y_i, z_i)$, with n degrees of freedom, subject to ideal olonom – scleronom (without any friction) constraints. These constraints can be defined by $3N - n$, functions of point coordinates, functionally independent and satisfying the relations:

$$\varphi_j(x_1, y_1, z_1, ..., x_N, y_N, z_N) = 0 \quad (j = 1,2,3,....,3N - n) \tag{1}$$

Because φ are independent functions, it results that the matrix $A(x_1, y_1, z_1, ..., x_N, y_N, z_N)$, defined by the relation:

$$A = \begin{vmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_1} & . & \frac{\partial\varphi_{3N-n}}{\partial x_1} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial\varphi_1}{\partial z_N} & \frac{\partial\varphi_2}{\partial z_N} & . & \frac{\partial\varphi_{3N-n}}{\partial z_N} \end{vmatrix} \tag{2}$$

having $3N$ rows and $3N - n$ columns, rank $r = 3N - n$. There is at least a determinant D_k with $3N - n$ rows and $3N - n$ columns, obtained by suppressing a number of n rows of the A matrix which is not identical null. Several such determinants, which satisfy the requirements, are likely to exist:

$$D_k(x_1, y_1, z_1, ..., x_N, y_N, z_N) \neq 0 \quad (k = 1,2,3,...) \tag{3}$$

Let us further consider a system of N active forces $\vec{F}_i(X_i, Y_i, Z_i)$, acting upon the material particles $A_i(x_i, y_i, z_i)$. The constraint forces act also on these particles. As such, due to the relation expressed through the equation $\varphi_j = 0$, the component constraint force will act upon the particle $A_{I,,}$ as follows:

$$\lambda_j \frac{\partial\varphi_j}{\partial x_i} \quad ; \quad \lambda_j \frac{\partial\varphi_j}{\partial y_i} \quad ; \quad \lambda_j \frac{\partial\varphi_j}{\partial z_i} \tag{4}$$

where by $\lambda_{j,}$ several parameters, undetermined for the time being, have been noted.

The system of forces acting upon the system of material particles is in equilibrium, if the resultants of the forces (active and constraint ones) acting upon each particle separately are null. The equations resulting for point A_i are as follows:

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$$X_i + \sum_{j=1}^{3N-n} \lambda_j \frac{\partial \varphi_j}{\partial x_i} = 0 \quad ; \quad Y_i + \sum_{j=1}^{3N-n} \lambda_j \frac{\partial \varphi_j}{\partial y_i} = 0 \quad ; \quad Z_i + \sum_{j=1}^{3N-n} \lambda_j \frac{\partial \varphi_j}{\partial z_i} = 0 \tag{5}$$

A number of $3N$ such equations are obtained for the whole system of material particles, to which a number of $3N - n$ equations are added (1), obtaining in all a number of $6N - n$ equations with $6N - n$ unknown quantities: $3N$ coordinates x_i, y_i, z_i and $3N - n$ parameters λ_j.

Introducing the column matrices f, λ and φ, defined by the relations:

$$\begin{aligned} f^T &= [X_1, Y_1, Z_1, ..., X_N, Y_N, Z_N] \\ \lambda^T &= [\lambda_1, \lambda_2, \lambda_3, ..., \lambda_{3N-n}] \\ \varphi^T &= [\varphi_1, \varphi_2, \varphi_3, ..., \varphi_{3N-n}] \end{aligned} \tag{6}$$

the equilibrium conditions for the forces acting upon the whole system of material particles are written in the form of a matrix, as follows:

$$f + A\lambda = 0 \quad ; \quad \varphi = 0 \tag{7}$$

There is the possibility to highly reduce the number of unknown notions, if from the $3N - 1$ equations (1) $3N - n$ coordinates, function of the other n are eliminated, or all Cartesian coordinates, more generally, function of the n generalized coordinates $q_1, q_2, \ldots q_n$, in the form:

$$x_i = x_i(q_1, ..., q_n) \quad ; \quad y_i = y_i(q_1, ..., q_n) \quad ; \quad z_i = z_i(q_1, ..., q_n) \qquad (i = 1,2,..., N) \tag{8}$$

In this case, the equations (1) are identically satisfied and it remains only the matrix equation:

$$f + A\lambda = 0 \tag{9}$$

Further on, in order to eliminate the λ_j parameters, we will see that (9) is a system of simultaneous linear algebra equations in λ_j and will apply the general theory of such equations, according to which such a system of equations has a solution, if and only if the A matrix of the system and the increased $A|f$ matrix obtained by the addition of a column with free terms to the A matrix, has the same rank, namely:

$$\text{rank } A = \text{rank } A|f \tag{10}$$

By writing the necessary requirements for imposing the condition (10), by canceling some determinants of the matrix $A|f$, we will obtain the required equations for the determination of the $q_1, q_2, \ldots q_n$ generalized coordinates and implicitly of the rest configuration of the given system of material particles under the action of the active forces $\vec{F}_i$.

Important remark: In the definition relation (2) of the matrix A, it is expressed as a function of *Cartesian coordinates* x_i, y_i, z_i $(i = 1,2,..., N)$ and has rank $3N - n$, at least one of the determinants D_k (relation 3), expressed as well function of the *Cartesian coordinates*, not being *identical* null. In the matrix equation (9), the matrix A is expressed as a function of the *generalized coordinates* $q_1, q_2, \ldots q_n$. It is possible for all D_k,, determinants which are not identical null when expressed as functions of the Cartesian coordinates, to be cancelled when they are expressed as functions of the generalized coordinates, which is a case when the rank of the matrix A of (9) is lower than $3N - n$.

The systems of material particles, where a change (specifically a decrease) of the rank of the matrix *A* takes place when the Cartesian coordinates are replaced as functions of the generalized coordinates, will be called CRITICAL SYSTEMS, and the systems of material particles where the rank of the matrix remains the same will be called NON-CRITICAL SYSTEMS.

It is to be mentioned that in the case of critical systems, the reactions or part of these remain undetermined.

Obviously, other methods of solving the equilibrium issue related to the system forces acting on a system of material particles (for instance the stiffening method) could also be applied within the field of classical theoretic mechanics.

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Analytical mechanics uses the virtual mechanical work principle. The virtual movements δx_i ; δy_i ; δz_i $(i = 1,2,...,N)$ are *infinitesimal and compatible with the constraints.* For reasons of the compatibility property, we can write for one of the functions (1) defining the constraints, the relation:

$$\delta\varphi_j = \sum_{i=1}^{N}\left(\frac{\partial\varphi_i}{\partial x_i}\delta x_i + \frac{\partial\varphi_i}{\partial y_i}\delta y_i + \frac{\partial\varphi_i}{\partial z_i}\delta z_i\right) = 0 \quad (j = 1,2,...,3N-n) \tag{11}$$

Taking into account the expression (4) of the components of some of the constraint forces from point A_i, the virtual mechanical work produced by this force, it results:

$$\lambda_j\left(\frac{\partial\varphi_j}{\partial x_i}\delta x_i + \frac{\partial\varphi_j}{\partial y_i}\delta y_i + \frac{\partial\varphi_j}{\partial z_i}\delta z_i\right) \tag{12}$$

Based on the virtual mechanical work of all constraint forces from all the material particle points of the system, it results:

$$\delta L = \sum_{j=1}^{3N-n}\lambda_j\sum_{i=1}^{N}\left(\frac{\partial\varphi_i}{\partial x_i}\delta x_i + \frac{\partial\varphi_i}{\partial y_i}\delta y_i + \frac{\partial\varphi_i}{\partial z_i}\delta z_i\right) = \sum_{j=1}^{3N-n}\lambda_j\delta\varphi_i \tag{13}$$

and on basis of the relation (11):

$$\delta L = 0 \tag{14}$$

a relation expressing in a first formulation the virtual mechanical work principle: "The virtual mechanical principle of the constraint forces of a system of material particles, subject to ideal constraint (without any friction) is null for any virtual movement compatible with the constraints.

Further on, multiplying the relations (5), respectively by $\delta x_i, \delta y_i$ $^o i$ δz_i and summing up all material particles of the system, we obtain:

$$\sum_{i=1}^{N}\left(X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i\right) + \sum_{j=1}^{3N-n}\lambda_j\sum_{i=1}^{N}\left(\frac{\partial\varphi_j}{\partial x_i}\delta x_i + \frac{\partial\varphi_j}{\partial y_i}\delta y_i + \frac{\partial\varphi_j}{\partial z_i}\delta z_i\right) = 0 \tag{15}$$

and taking into account the relations (13) and (14), it results:

$$\sum_{i=1}^{N}\left(X_i\delta x_i + Y_i\delta y_i + Z_i\delta z_i\right) = 0 \tag{16}$$

a relation expressing in a second formulation the virtual mechanical work principle: *The virtual mechanical work principle of the active forces acting upon a system of material particles subject to ideal constraints (without any friction) is null for any virtual movement compatible with the constraints*.

Comments: Being infinitesimal sizes, the virtual movements can be regarded as variables tending towards zero. At the same time, we can see that the product $\lambda_j\delta\varphi_i$ tends towards zero, only if λ_j tends to a finite value, that is if the equation system from where it originates is a *consistent one*, that is if the general theory of simultaneous linear algebra equations leading to the condition (10) is observed. And as a matter of fact, this condition is disregarded in the formulation and application of the virtual mechanical work principle. It is therefore expected that for this reason, the method of virtual mechanical work should lead to more solutions than the methods of the classical theoretic mechanics and namely to solutions where reactions tend towards infinite, unacceptable, having originated from inconsistent simultaneous linear systems.

The principle of the virtual mechanical work leads to the same solutions like the methods of classical theoretical mechanics, only in case of the non-critical systems.

An example will actually clarify the issue. Let us consider two material particles, of equal weights forced to remain on a circle with the radius r and at a distance $2d < 2r$. The angle θ corresponding to the rest configuration is requested (fig. 1).

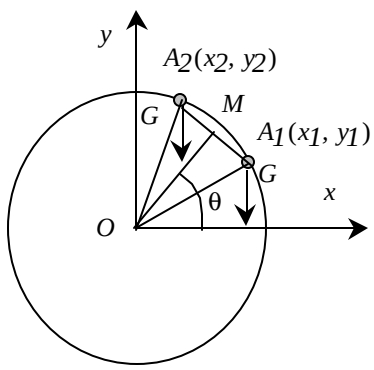

Fig. 1

Let us consider $A_1(x_1, y_1)$ and $A_2(x_2, y_2)$. The connection equations are written:

$$x_1^2 + y_1^2 - r^2 = 0$$
$$x_2^2 + y_2^2 - r^2 = 0$$
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 - d^2 = 0$$

Parametrically, function of the angle θ, considered as a generalized coordinate, the coordinates x_1, y_1, x_2, y_2 is expressed through the relations:

$$x_1 = \sqrt{r^2 - d^2}\cos\theta + d\sin\theta \quad ; \quad y_1 = \sqrt{r^2 - d^2}\sin\theta - d\cos\theta$$
$$x_2 = \sqrt{r^2 - d^2}\cos\theta - d\sin\theta \quad ; \quad y_2 = \sqrt{r^2 - d^2}\sin\theta + d\cos\theta$$

We will use the stiffening method of the classical mechanics, by writing the equation of moments related to *O for the whole system.* We will obtain:

$$-Gx_1 - Gx_2 = 0 \quad ; \quad x_1 + x_2 = 0 \quad ; \quad 2\sqrt{r^2 - d^2}\cos\theta = 0 \quad ; \quad \theta = \pm\frac{\pi}{2}$$

By using the principle of the virtual mechanical work, we obtain:

$$-G\delta y_1 - G\delta y_2 = 0 \quad ; \quad \delta(y_1 + y_2) = 0 \quad ; \quad \delta\left[2\sqrt{r^2 - d^2}\sin\theta\right] = 0 \quad ; \quad 2\sqrt{r^2 - d^2}\cos\theta \cdot \delta\theta = 0 \quad ; \quad \theta = \pm\frac{\pi}{2}$$

The results coincide. There are two rest configurations, as shown in figure 2.

Let us further solve the same problem in the case $2d = 2r$.

In this case, the equations will write:

$$x_1^2 + y_1^2 - r^2 = 0 \quad ; \quad x_2^2 + y_2^2 - r^2 = 0 \quad ; \quad (x_1 - x_2)^2 + (y_1 - y_2)^2 - 4r^2 = 0$$

Parametrically, function of the angle θ (fig.3), considered as a generalized coordinate, the Cartesian coordinates x_1, y_1, x_2, y_2 are expressed through the relations:

$$x_1 = r\cos\theta \quad ; \quad y_1 = r\sin\theta \quad ; \quad x_2 = -r\cos\theta \quad ; \quad y_2 = -r\sin\theta$$

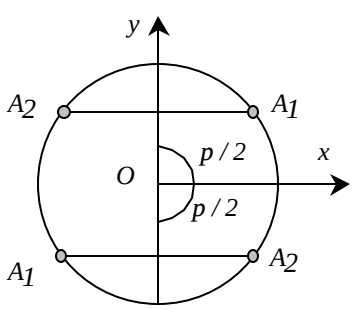

Fig. 2

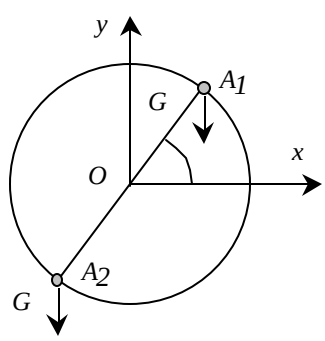

Fig. 3

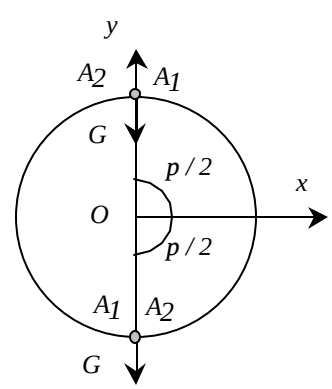

Fig. 4

By using the stiffening method of the classical mechanics, we will write the equation of projection on the normal in case of A_1A_2 for the whole system. We will obtain:

$$G\cos\theta + G\cos\theta = 0 \quad ; \quad 2G\cos\theta = 0 \quad ; \quad \theta = \pm\frac{\pi}{2}$$

By using the principle of the virtual mechanical work, we will obtain:

$$-G\,\delta y_1 - G\,\delta y_2 = 0 \quad ; \quad \delta(r\sin\theta) + \delta(-r\sin\theta) = 0 \quad ; \quad 0 = 0$$

Consequently, the equilibrium is achieved for any angle θ.

The two results do not coincide. The correct solution is the one given in the classical theoretical mechanics and is shown in figure 4.

The lack of coincidence is motivated by the fact that in the case $2d = 2r$, the system formed by the two particles becomes critical:

$$A(x_1, y_1, x_2, y_2) = \begin{vmatrix} 2x_1 & 0 & 2(x_1 - x_2) \\ 2y_1 & 0 & 2(y_1 - y_2) \\ 0 & 2x_2 & -2(x_1 - x_2) \\ 0 & 2y_2 & -2(y_1 - y_2) \end{vmatrix} \quad ; \quad A(\theta) = \begin{vmatrix} 2r\cos\theta & 0 & 4r\cos\theta \\ 2r\sin\theta & 0 & 4r\sin\theta \\ 0 & -2r\cos\theta & -4r\cos\theta \\ 0 & -2r\sin\theta & -4r\sin\theta \end{vmatrix}$$

Indeed, the matrix A, expressed as a function of the Cartesian coordinates x_1, y_1, x_2, y_2 has the rank 3, the determinants of the third rank being $8x_2(x_1y_2 - x_2y_1)$ and $8y_2(x_1y_2 - x_2y_1)$, that is no one being **identical null**, whereas the matrix A, expressed as a function of the generalized coordinate θ has the rank 2, the determinants of the third order being **all indentical null** and there existing determinants of the second order different from zero.

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